Perfect Fréchet Spaces

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A perfect Fréchet Space is a perfect sequence space, \( \lambda \), which is a Fréchet space when equipped with the strong topology from its Köthe dual (or cross), \( \lambda^\times \). According to [2], this is equivalent to saying that \( \lambda \) is a Fréchet space with a weak Schauder basis that is both bounded multiplier and boundedly complete. Various authors have discussed the construction of perfect Fréchet spaces by defining an object called an echelon space of order \( p \) ([4], [3], [1]). It is the purpose of this paper to generalize the construction of echelon spaces and show that one obtains all perfect Fréchet spaces in this way.

All of the notation and undefined terminology are explained in [5]. We list the less familiar instances along with the few exceptions, below.

By a sequence space we mean a vector space of sequences of complex numbers which includes all finite sequences. If \( \lambda \) is a sequence space, we define its cross, \( \lambda^\times \), by

\[
\lambda^\times = \left\{ u = (u_i) \left| \sum_{i=1}^{\infty} |x_i u_i| < \infty \right. \quad \text{for all } x = (x_i) \in \lambda \right\}.
\]

We say that \( \lambda \) is perfect if \( \lambda = \lambda^{\times \times} \). We say that \( \lambda \) is normal if \( x \in \lambda, |y| \leq |x| \) for all \( i \) implies \( y \in \lambda \). The pair \( \langle \lambda, \lambda^\times \rangle \) is a dual system [5, p. 88] and if \( A \subseteq \lambda \), we define the polar, \( A^0 \), with respect to \( \langle \lambda, \lambda^\times \rangle \) by

\[
A^0 = \left\{ u \in \lambda^\times \left| \langle x, u \rangle = \left| \sum_{i=1}^{\infty} x_i u_i \right| \leq 1 \quad \text{for all } x \in A \right. \right\}.
\]

If \( \lambda \) is perfect then weak and strong bounded sets in the dual system coincide ([5], p. 416) so in this case we shall refer unambiguously to bounded sets.

We shall use subscripts to index the co-ordinates of a sequence and superscripts to index sequences. In particular, \( e^n \) will refer to the sequence which is zero at all co-ordinates but the \( n \) th co-ordinate where it is one.

If \( x = (x_i) \) then we define the \( m \) th section, \( x^m \), to be the sequence whose first \( m \) co-ordinates agree with \( x \) and its remaining co-ordinates are zero. We write \( |x| = (|x_i|) \) and \( 1/x = (1/x_i) \) provided the terms of \( x \) are not zero. If \( a = (a_i) \) is a sequence and \( \lambda \) a sequence space we define the diagonal transformation by

\[
a \lambda = \{(a_i x_i) | x = (x_i) \in \lambda \}.
\]

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If $S$ is an arbitrary set of sequences, we define $aS$ similarly. Some of the more important sequence spaces are $\ell^\infty$, the space of all finite sequences; $\ell^p$ (1 ≤ $p$ ≤ $\infty$) the space of all $p$-summable sequences, and $c_0$, the space of all null sequences.

Let $E, F$ be vector spaces such that $(E, F)$ is a dual system. Let $\Sigma$ be a saturated class of weakly bounded subsets of $F$ whose union is $F$. We shall denote the topology, on $E$, of uniform convergence on elements of $\Sigma$ by $\mathcal{T}_\Sigma(F, E)$ (or $\mathcal{T}_\Sigma(F)$ or $\mathcal{T}_\Sigma$ if the context prevent misunderstanding) and refer to $\mathcal{T}_\Sigma$ as a $\langle E, F \rangle$-topology.

1. Preliminaries

If $\lambda$ is a perfect space and $\mathcal{T}$ is a $\langle \lambda, \lambda^\times \rangle$-topology, then it is possible for $\lambda[\mathcal{T}]$ to be barrelled and at the same time, for $\mathcal{T}_b(\lambda^\times)$ to be different from the strong topology (e.g., $\lambda = \ell^\infty$). If $\lambda[\mathcal{T}]$ is a Montel space, the situation is much simpler.

Lemma 1. Let $\lambda$ be perfect and $\mathcal{T}$ a $\langle \lambda, \lambda^\times \rangle$-topology such that $\lambda[\mathcal{T}]$ is a Montel space. Then $\mathcal{T}_b(\lambda^\times) = \mathcal{T}_b(\lambda^\times)$.

Proof. Since $\lambda[\mathcal{T}]$ is barrelled, $\mathcal{T} = \mathcal{T}_b(\lambda^\times)$. Every $\mathcal{T}_b(\lambda^\times)$-compact set is $\mathcal{T}_b(\lambda^\times)$-closed and $\mathcal{T}_b(\lambda^\times)$-bounded. Hence since $\lambda$ is perfect, it is $\mathcal{T}_b(\lambda^\times)$-bounded and therefore $\mathcal{T}_b(\lambda^\times)$-compact since $\lambda[\mathcal{T}]$ is a Montel space. Hence $\lambda[\mathcal{T}_b(\lambda^\times)]$ is barrelled [5, p. 420] and the result follows.

In view of the previous result, we may speak without ambiguity of a perfect space, $\lambda$, being a Montel space in which case the topological dual is $\lambda^\times$.

Let $J$ be a subsequence (not finite) of the set $\mathbb{N}$ of natural numbers. If $\lambda$ is a sequence space, we define

$$\lambda_J = \{x = (x_j) \mid \exists y \in \lambda \text{ with } y_{n_j} = x_j \text{ for all } n_j \in J\}.$$  

We call $\lambda_J$ the stepspace of $\lambda$ corresponding to $J$. If $x \in \lambda_J$, then the canonical pre-image of $x$ is the sequence $y$ which agrees with $x$ on the indices in $J$ and is zero elsewhere. If $J_1, J_2$ are disjoint subsequences of $\mathbb{N}$ whose union is $\mathbb{N}$, we say that $\lambda_{J_1}, \lambda_{J_2}$ are complementary stepspaces.

Lemma 2. Let $\lambda$ be a normal sequence space and $\lambda_J$ a stepspace. Then,

i) $\lambda_J$ is normal.

ii) $(\lambda_J)^\times = (\lambda^\times)_J$.

iii) If $\lambda$ is perfect, so is $\lambda_J$.

iv) If $B \subseteq \lambda_J$, then $B$ is weakly (resp. strongly) bounded if and only if its canonical pre-image is a weakly (resp. strongly) bounded subset of $\lambda$.

v) If $\lambda$ is a perfect Montel space so is $\lambda_J$.

vi) If $\lambda[\mathcal{T}_b(\lambda^\times)]$ is semi-reflexive (resp. reflexive), so is $\lambda_J[\mathcal{T}_b(\lambda^\times)]$.

Proof. i), ii), iii), iv) are straightforward and v), vi) follow easily from the first four and characterizations of these properties given by Köthe [5, §30,7].

Theorem 1. If $\lambda$ is a perfect space such that $l^1 \subseteq \lambda \subseteq l^\infty$, then $\lambda$ is not a perfect Montel space.

Proof. First suppose $l^1 \subseteq \lambda \subseteq c_0$. Then $(e^n)$ is a $\mathcal{T}_\lambda(\lambda, \lambda^\times)$-null sequence in $\lambda^\times$ and also a bounded subset of $\lambda$. But $\langle \langle e^n, e^m \rangle \rangle_n$ is not uniformly convergent to zero with respect to $m$, so $\lambda^\times$ and hence $\lambda$ is not a perfect Montel space.