An Extension of Schütte's Klammersymbols*

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Introduction

A function $\varphi$ which maps the second number class into itself is called a normal function if $\varphi$ is strictly monotonic and continuous. It is well known [1, pp. 37--38] that every normal function $\varphi$ has fixed points and that the ordering function of these fixed points, written $\varphi'$, is again a normal function.

Let $\Omega$ denote the first uncountable ordinal. In [6] a family $\{\varphi_c\}_{c<\Omega}$ of normal functions is defined with the property that $\varphi_0 = \varphi$ and $\varphi_\alpha$ is the ordering function of the fixed points of $\varphi_\beta$ for all $\beta$ such that $0 \leq \beta < \alpha$. It can easily be shown that if $\varphi_0(0) \neq 0$ then $\{x : \varphi_\alpha(x) = x \text{ for all } \alpha < \Omega\}$ is empty. Hence one cannot obtain a normal function $\varphi_\Omega$ by enumerating the fixed points of all $\varphi_\beta$ such that $\beta < \Omega$.

BACHMANN [2] solves the problem of how to pass to a normal function $\varphi_\Omega$: indeed, he obtains a family $\{\varphi_c\}_{c<\beta}$ of normal functions where $\beta$ is a particular ordinal of the third number class. Let $\varepsilon_{\Omega+1}$ be the first epsilon number greater than $\Omega$ or, equivalently, the first fixed point of the function $\Omega x$. In this paper we shall be concerned with BACHMANN's family $\{\varphi_c\}$ restricted to $c < \varepsilon_{\Omega+1}$ and with certain equivalent families.

In §1 we prove some lemmas concerning BACHMANN's distinguished sequences which will enable us to construct our family of normal functions in §2.

We prove the Recursion, Representation and Closure theorems in §3. These theorems, when easily modified so as to apply to the $\varphi_\Omega$ hierarchy in §4, form the basis for a system of ordinal notations generalizing SCHÜTTE's [5] notations.

§1. Some Lemmas on Distinguished Sequences

We know that given any ordinal $\gamma \geq 2$ then every ordinal $\alpha \geq 1$ can be written uniquely in the form

\begin{equation}
\alpha = \sum_{i=1}^{n} \gamma^{\alpha_i} a_i
\end{equation}

with $\alpha \geq \alpha_n > \cdots > \alpha_1 \geq 0$, $1 \leq a_i < \gamma$ for $i = 1, \ldots, n$ where $1 \leq n < \omega$.

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If $\alpha$ is represented in form (1), we say that it is expressed in (Cantor) normal form to the base $\gamma$. The following definition of the distinguished sequence associated with $\alpha$ is obtained from the representation of $\alpha$ in normal form to the base $\Omega$.

**Definition 1.** To every limit ordinal $\alpha < \varepsilon_{\Omega+1}$ we associate a distinguished sequence $\{x_c\}$ whose type is a limit number $\tau_\alpha \leq \Omega$ so that

$$\lim_{c < \tau_\alpha} x_c = \alpha$$

as follows:

1.1. If $\alpha = \Omega^{x_1} + 1$ then $x_c = \Omega^{x_1} c$ for all $c$ such that $0 < c < \Omega$.

1.2. If $\alpha = \Omega^{x_1}$ where $\alpha_1$ is a limit ordinal $< \varepsilon_{\Omega+1}$ then $x_c = \Omega^{\gamma_c}$ where $\{\gamma_c\}$ is the distinguished sequence for $\alpha_1$.

1.3. If $\alpha = \Omega^{x_1} a_1$ where $a_1$ is a limit ordinal $< \Omega$ then $x_c = \Omega^{x_1} c$ for all $c$ such that $0 < c < a_1$.

1.4. If $\alpha = \Omega^{x_1} (a_1 + 1)$ with $a_1 + 1 < \Omega$ then $x_c = \Omega^{x_1} a_1 + y_c$ where $\{y_c\}$ is the distinguished sequence for $\Omega^{x_1}$.

1.5. If $\alpha = \sum_{i=1}^{n} \Omega^{x_i} a_i$ is written in normal form then $x_c = \sum_{i=2}^{n} \Omega^{x_i} a_i + y_c$ where $\{y_c\}$ is the distinguished sequence for $\Omega^{x_1} a_1$.

Definition 1 is just BACHMANN’s [2, pp. 121—122] notion of distinguished sequence.

If $\alpha$ is a limit ordinal $< \varepsilon_{\Omega+1}$ and $\{x_c\}$ is the distinguished sequence such that $\lim_{c < \tau_\alpha} x_c = \alpha$ then we call $\{x_c\}$ the sequence associated with $\alpha$. The type of the sequence $\{x_c\}$ is $\tau_\alpha$.

We find it convenient to introduce the following definitions.

**Definition 2.** Let $0 \leq \alpha < \varepsilon_{\Omega+1}$. We define the relation $C(x, \alpha)$ as follows:

2.1. If $\alpha = 0$ then $C(x, \alpha)$ iff $x = 0$.

2.2. If $\alpha \neq 0$ let $\alpha = \sum_{i=1}^{n} \Omega^{x_i} a_i$ be the normal form (1) of $\alpha$, then $C(x, \alpha)$ iff $x = a_i$ or $C(x, \alpha_i)$ for some $i$.

We read $C(x, \alpha)$ as “$x$ is a constituent of $\alpha$”.

**Definition 3.** Let $0 \leq \alpha < \varepsilon_{\Omega+1}$. We define the relation $P(x, \alpha, a)$ as follows:

$P(x, \alpha, a)$ iff $x = a$ or $C(x, \alpha)$.

We read $P(x, \alpha, a)$ as “$x$ is a part of $(\alpha, a)$”.

**Definition 4.** Let $\alpha = \sum_{i=1}^{n} \Omega^{x_i} a_i$ be expressed in normal form (1). We define the distinguished part of $(\alpha, a)$, written $D(\alpha, a)$, as follows:

4.1. $(0, 0)$ has no distinguished parts.

4.2. If $a \neq 0$ then $D(\alpha, a) = a$.

4.3. If $a = 0$ and $a_1 \neq 1$ then $D(\alpha, a) = a_1$.

4.4. If $a = 0$ and $\alpha_1 = 0$ then $D(\alpha, a) = a_1$.

4.5. If $a = 0$, $\alpha_1 \neq 0$ and $a_1 = 1$ then $D(\alpha, a) = D(\alpha_1, a)$. 