Non-analytic Hypersurfaces in $C^n$*

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The purpose of this paper is to give examples of strictly pseudoconvex $C^\infty$ real hypersurfaces in $C^n$ ($n \geq 2$) which are not locally CR equivalent to analytic hypersurfaces. The following ideas motivate the search for such examples.

Let $f : D \rightarrow D'$ be a biholomorphism of bounded domains in $C^n$. Assume $\partial D, \partial D'$ are smooth and that $f$ extends to a smooth map $\partial D \rightarrow \partial D'$. Let $p \in \partial D$. If the Levi form of $\partial D$ at $p$ is non-degenerate but indefinite, by the Lewy extension theorem ([2], Theorem 2.6.13), $f$ extends biholomorphically to a full neighborhood (in $C^n$) of $p$. As a consequence, if $\partial D$ is analytic near $p$ (i.e., given as the zeroes of a real analytic function in a neighborhood of $p$), $\partial D'$ is analytic near $f(p)$. In other words, $f$ extends as an analytic map, not merely as a smooth map. If $\partial D$ is strictly pseudoconvex, however, $f$ may indeed extend only to a smooth map of the boundaries. For example, let $g(z)$ be a biholomorphic map between the unit disk in $C$ and a domain in $C$ with smooth non-analytic boundary. Choose such a $g(z)$ which extends to a smooth map of the boundaries. Then $G : (z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_{n-1}, g(z_n))$ is a biholomorphic map of the unit ball in $C^n$. $G$ extends to a smooth map from the unit sphere to some non-analytic hypersurface. The question arises, is every strictly pseudoconvex $C^\infty$ hypersurface locally equivalent in this fashion to an analytic hypersurface?

This is true in the lowest dimension, the case of arcs in $C$. Any (sufficiently small piece of) arc is part of the boundary of a simply connected domain. We obtain an equivalence of the arc with the unit circle by applying the Riemann mapping theorem, mapping the domain onto the unit disk.

However, this is not true in higher dimensions. In this paper we construct, using the geometric invariants developed by S. S. Chern and J. K. Moser [1], examples of strictly pseudoconvex hypersurfaces in $C^n$ which are not locally equivalent to analytic hypersurfaces.

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Example. Let \( \varphi : \mathbb{R} \to \mathbb{R} \) be a non-negative \( C^\infty \) function such that \( \varphi(u) = 0 \) if \( u \leq 0 \), \( \varphi(u) = 1 \) if \( u \geq 1 \). Consider the hypersurface \( M \subseteq \mathbb{C}^{n+1} = \{(z_1, \ldots, z_n, u+iv)\} \) given by

\[
v = \langle z, z \rangle + \varepsilon \varphi(u)(z_1^4 \bar{z}_1^2 + z_2^2 \bar{z}_2^4) \tag{1}
\]

where \( \langle z, z \rangle = \sum_{j=1}^{n} z_j \bar{z}_j \), and \( \varepsilon \in \mathbb{R} - \{0\} \). We shall show that for \( |\varepsilon| \) sufficiently small, there exists a point of \( M \) around which \( M \) is not locally equivalent to an analytic hypersurface.

We summarize the results we use. (We follow the notation of \cite{1}.)

Given a non-degenerate real hypersurface \( M \subseteq \mathbb{C}^{n+1} \), one can construct a bundle \( p : Y \to M \) with a Cartan connection \( \pi \) (which can be viewed as a matrix of one-forms on \( Y \)). The curvature of the connection can be expressed using curvature functions on \( Y \). These are all intrinsic in the following sense. Suppose \( M \) and \( M' \) are two equivalent hypersurfaces, say by a map \( f : M \to M' \). Then there exists a map \( F : Y \to Y' \) covering \( f \) so that \( F^* \pi' = \pi \). (This implies that the curvature functions on \( Y \) are just those on \( Y' \) pulled back by \( F \).) These invariants \( (Y, \pi) \) are complete in the sense that any such map \( F \) comes from some \( f \). Hence, \( Y \) and \( \pi \) determine the structure of \( M \). For example, the curvature of \( \pi \) vanishes if and only if \( M \) is locally equivalent to the hyperquadric \( Q = \{(z_1, \ldots, z_n, u+iv) \in \mathbb{C}^{n+1} | v = \langle z, z \rangle \} \), with \( \langle z, z \rangle = \sum_{j,k=1}^{n} g_{jk} z_j \bar{z}_k \), with \( (g_{jk}) \) a non-singular hermitian matrix. Such \( M \) are called flat.

If \( M \) is an analytic hypersurface, \( Y \) and \( \pi \), and hence the curvature of \( \pi \), are all analytic. Moreover, \( M \) has a normal form. We can find a biholomorphic map which (locally) takes \( M \) into the locus

\[
v = \langle z, z \rangle + \sum_{\min(k,l) \geq 2} F_{kl}, \tag{2}
\]

where \( \langle z, z \rangle \) is as just above, \( F_{kl} = F_{kl}(z, \bar{z}, u) \) is a polynomial in \( z \) and \( \bar{z} \) (with coefficients analytic functions of \( u \)) homogeneous of degree \( k \) in \( z \) and degree \( l \) in \( \bar{z} \). Moreover, \( F_{22}, F_{32} = F_{32}, F_{33} \) satisfy certain trace conditions. If \( n = 1 \) (a hypersurface in \( \mathbb{C}^2 \)), the trace conditions are \( F_{22} = F_{32} = F_{33} = 0 \). This normal form is unique up to a transformation preserving the origin and the hyperquadric \( \{v = \langle z, z \rangle \} \). Thus \( M \) is locally equivalent to the hyperquadric if and only if its normal form is \( v = \langle z, z \rangle \).

We shall use the following fact. Let \( M \) be a smooth manifold, and let \( g \) be a smooth real-valued function on \( M \). We call \( g \) locally analytic if in a neighborhood of each point of \( M \) there exists a coordinate system in which \( g \) is an analytic function of the coordinates. One can then show, just as in the case of analytic functions on an analytic manifold, that if a locally analytic function vanishes on an open set, it vanishes everywhere. Note that if a hypersurface is everywhere locally equivalent to an analytic hypersurface, then \( \pi \) and its curvature are locally analytic.

Consider now our example \( M = \{v = \langle z, z \rangle + \varepsilon \varphi(u)(z_1^4 \bar{z}_1^2 z_2^2 \bar{z}_2^4)\} \). Since all the derivatives of \( M \) are bounded, by taking \( |\varepsilon| \) small we can make \( M \) strictly pseudoconvex (for \( \|z\|^2 = \langle z, z \rangle \) small). If \( u < 0 \) or \( u > 1 \), \( M \) is analytic and actually in