Closures of Supplementary Vector Subspaces

SHOURO KASAHARA

1. Introduction

In this paper, we shall be concerned with the following problem: Under what condition does there exist a locally convex Hausdorff topology $\tau$ on a vector space $E$ which makes a given vector subspace $M$ of $E$ dense?\(^1\)

We shall consider this problem under the additional restrictions on the topology $\tau$:

1. $\tau$ makes an algebraic supplement $N$ of $M$ closed; or
2. $\tau$ makes an algebraic supplement $N$ of $M$ dense;

in connection with or without given locally convex Hausdorff topologies on $M$ and $N$.

The problem is treated mainly under the restriction (1), to which sections 4, 5 and 6 are devoted. These three sections correspond to the topologies given on $M$ and $N$, that is, weak topologies on them; locally convex topology on $M$ and weak topology on $N$; and locally convex topology on $M$, respectively. Section 7 is concerned with the case where the restriction (2) is imposed together with the restriction that the induced topology of $\tau$ on $M$ is coarser than a given weak topology on $M$. In the last Section 8, we consider the problem without any restrictions on induced topologies on $M$ and $N$. Section 3 is devoted to discuss, as preliminaries, the dimension and the codimension of a vector subspace of $E^*$, the algebraic dual of $E$, which constitutes a separated dual system together with $E$.

Section 7 contains a theorem which states the existence of a locally convex Hausdorff topology on $E$ which makes both $M$ and $N$ closed, and induces on $M$ and $N$ given topologies on $M$ and $N$ respectively. This theorem permits us, by combining another result, to derive the condition which ensures the existence of locally convex Hausdorff topology on $E$ under which the closure of $M$ coincides with a given vector subspace of $E$. However, it is very easy to derive these combining results, so that we do not set out them.

The author is indebted to Professor T. ISHIHARA for suggesting the problem, and would like to thank him for his helpful advice and encouragement. The author would like to thank also Professor K. KUNUGI for his helpful criticism and encouragement.

\(^1\) The contents of the present paper are partly contained in [2] and [3]. However, I of [3] contains several errors, and the present paper is a revised form of them.
2. Terminology and notations

We shall be concerned exclusively with vector spaces over the field $C$ of complex numbers. But every result obtained in what follows remains valid for vector spaces over the field of real numbers.

Let $E$ be a vector space, and let $M$ be a vector subspace of $E$. $M$ is called non-trivial if $M \neq \{0\}$. By a supplement of $M$ in $E$, we mean an algebraic supplement of $M$ in $E$, that is, a vector subspace $N$ of $E$ such that $M + N = E$ and $M \cap N = \{0\}$. By a base of $E$, we mean always a Hamel base of $E$. We denote by $\dim(E)$ the dimension of $E$, i.e. the cardinal number of a base of $E$, and by $\text{codim}_E(M)$ the codimension of $M$ in $E$. The algebraic dual of $E$ is denoted by $E^*$. Whenever a topology is assigned to $E$, we denote by $E'$ the dual of $E$ for the topology. A locally convex topology $\tau$ on $E$ is called a $d_M$ topology (resp. $c_M$ topology) if $M$ is dense in $E$ (resp. $M$ is closed) for the topology $\tau$. Further, by saying a topology $\tau$ is an $x_My_N$ topology, where $x, y$ are $d$ or $c$ and $M \cap N = \{0\}$, we mean that $\tau$ is an $x_M$ topology and a $y_N$ topology at the same time. Let $\tau$ be a topology on $E$. The induced topology of $\tau$ on $M$ is denoted by $\tau|_M$, and the restriction to $M$ of a mapping $u$, defined on $E$, is denoted by $u|_M$. Let $\tau_1$ and $\tau_2$ be two topologies on $E$; by $\tau_1 \supset \tau_2$ or $\tau_2 \subseteq \tau_1$, we mean $\tau_1$ is finer than $\tau_2$.

A pair $(E, E')$ of a vector space $E$ and a vector subspace $E'$ of $E^*$ is called a dual system. A dual system $(E, E')$ is said to be separated if for each non-zero element $x$ of $E$, there exists an $x'$ in $E'$ such that $\langle x, x' \rangle \neq 0$. Let $(E, E')$ be a dual system; the polar $A^c$ of a subset $A$ of $E$ (resp. $E'$) is the set of all $x' \in E'$ (resp. of all $x \in E$) such that $|\langle x, x' \rangle| \leq 1$ for every $x \in A$ (resp. for every $x' \in A$).

We denote by $\mathcal{F}(S, C)$ the vector space consisting of all complex valued functions on a set $S$, and by $\mathcal{B}(S, C)$ that of all complex valued bounded functions on $S$.

The symbol $\aleph$ denotes the cardinal number of the continuum, and $I$ the set of all positive integers.

3. Dimension and codimension of the dual

The purpose of this section is to state several propositions which we shall need in the following sections.

It is known\(^2\) that the dimension of the vector space $\mathcal{F}(S, C)$ is $2^m$ if the cardinal number $m$ of the set $S$ is infinite, or equivalently, that the dimension of the algebraic dual of an infinite dimensional vector space $E$ is $2^{\dim(E)}$.

We begin with a lemma which is a consequence of this fact.

**Lemma 3.1.** Let $E$ be an infinite dimensional vector space.

1° If a dual system $(E, E')$ is separated, then $\dim(E) \leq 2^{\dim(E')}$ and $\text{dim}(E') \leq 2^{\dim(E)}$.

2° For every cardinal number $m$ with $\dim(E) \leq 2^m$ and $m \leq 2^{\dim(E)}$, there exists a vector subspace $E'$ of $E^*$ such that the dual system $(E, E')$ is separated and $\dim(E') = m$.

\(^2\) See Köthe [4] or Mackey [5]; a direct proof of this fact is given in Köthe [4].