A New Generalization of the Schauder Fixed Point Theorem

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Introduction

Let $X$ be a Banach space, $K$ a compact convex subset of $X$. The well-known fixed point theorem of SCHAUDER [15] states that every continuous mapping $f$ of $K$ into $K$ has a fixed point. The generalization by TYCHONOFF [16] extends this result to the case in which $X$ is replaced by a general locally convex topological vector space $E$. TYCHONOFF's theorem contains as a special case the earlier result in SCHAUDER [15] asserting the existence of a fixed point for each weakly continuous self-mapping $f$ of a weakly compact convex subset $C$ of a Banach space $X$.

It is our object in the present paper to prove a new generalization of the SCHAUDER and TYCHONOFF fixed point theorems. This generalization includes the SCHAUDER and TYCHONOFF theorems and has two interesting general features:

(1) It does not seem to be obtainable by any easy direct method from the corresponding result in the finite dimensional case;

(2) It follows from an argument which uses the conjugate space $E^*$ of the locally convex topological vector space $E$.

We distinguish the results discussed here from previous generalizations of the SCHAUDER theorem published by the writer (BROWDER [1], [2], [3], [4], [5]) which center around the concept of asymptotic fixed point theorems and of deformations of non-compact mappings.

Before proceeding to the statement of our general results, we introduce the following definition:

**Definition 1.** Let $C$ be a closed convex subset of a locally convex topological vector space $E$. Then a point $x$ of $C$ is said to lie in $\delta(C)$ if there exists a finite dimensional subspace $F$ of $E$ such that $x$ lies in the boundary of $C \cap F$.

Our basic results are contained in the following two theorems:

**Theorem 1.** Let $E$ be a locally convex topological vector space, $K$ a compact convex subset of $E$, $f$ a continuous mapping of $K$ into $E$. Suppose that for each $u$ in $\delta(K)$, there exists an element $v$ of $K$ and a real number $\lambda$ with $\lambda > 0$, (both depending upon $u$), such that

$$ f(u) - u = \lambda (v - u). $$
Then \( f \) has a fixed point in \( K \).

**Theorem 2.** Let \( E \) be a locally convex topological vector space, \( K \) a compact convex subset of \( E \), \( f \) a continuous mapping of \( K \) into \( E \). Suppose that for each \( u \) in \( \delta(K) \), there exists an element \( v \) of \( K \) and a real number \( \lambda \) with \( \lambda < 0 \), (both depending upon \( u \)), such that

\[
f(u) - u = \lambda(v - u).
\]

Then \( f \) has a fixed point in \( K \).

**Remark 1.** If \( f \) satisfies the hypotheses of the Schauder or Tychonoff theorem so that \( f \) maps \( K \) into \( K \), we may take \( v = f(u) \) for every \( u \) and set \( \lambda = 1 \), and obtain the Tychonoff fixed point theorem as a special case of Theorem 1. On the other hand, in either Theorem 1 or 2, if we can choose \( v \) for the equation (1) for every \( u \) in \( K \) and if \( v \) can be chosen continuously in \( u \), then we may apply the Tychonoff theorem to the mapping \( v = S(u) \) and obtain a fixed point for \( S \) in \( K \) which is also a fixed point for \( f \) in \( K \). However, there is nothing in the hypothesis of Theorems 1 and 2 which insures that such a continuous choice of \( v \) is possible.

**Remark 2.** For a restricted class of convex sets in Banach spaces, the result of Theorem 2 for outward mappings \( f \) was obtained in a long direct argument by B. Halpern in his U.C.L.A. Ph.D. thesis in 1965 (unpublished).

**Remark 3.** Special cases of Theorems 1 and 2 which are important for application are those in which \( E \) is a Banach space \( X \) in either its strong, its weak, or, for \( E = X^* \), its weak* topologies.

Theorems 1 and 2 are obtained by a specialization of a simple result concerning mappings of \( E \) into its conjugate space \( E^* \). Here \( E^* \) is the topological vector space of continuous linear functionals on \( E \) topologized in the usual way (Kôthe [13]), and we denote the pairing between \( w \) in \( E^* \) and \( u \) in \( E \) by \( (w, u) \).

This result is the following:

**Theorem 3.** Let \( E \) be a locally convex topological vector space, \( K \) a compact convex subset of \( E \), \( T \) a continuous mapping of \( K \) into \( E^* \). Then there exists an element \( u_0 \) of \( K \) such that

\[
(T(u_0), u - u_0) \geq 0
\]

for all \( u \) in \( K \).

As a convenient specialization of Theorem 3, we have:

**Theorem 4.** Let \( E \) be a locally convex topological vector space, \( K \) a compact convex subset of \( E \), \( f \) a continuous mapping of \( K \) into \( E \). Let \( R \) be a continuous mapping of the set \( (I - f)(K) \) into \( E \), \( (I = \text{the identity map}) \). Then there exists an element \( u_0 \) of \( K \) such that

\[
(R(u_0 - f(u_0)), u - u_0) \geq 0
\]

for all \( u \) in \( K \).

In Section 1, we give the simple proof of Theorem 3 using the Brouwer fixed point theorem. Because of its simplicity, we give it here for the sake of completeness although it is a special case of Proposition 1 of Browder [9], [10]. For the case of finite-dimensional spaces, the result was first obtained by a