Quadratic Relations for Generators of Units in the Modular Function Field

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In [KL I] we found it convenient for the diophantine applications to work with differences and quotients of the Weierstrass elliptic functions in order to obtain units in the modular function field. For the multiplicative theory of [KL II], it was more convenient to deal with the Siegel functions (quotients of Klein forms). We shall now investigate the relations between these, and determine when a product of Klein forms is a modular function of the given level. At the same time, it is also natural to express modular functions whose divisor is at infinity as a product of Weierstrass \( \sigma \)-functions which differ from the Klein forms by an exponential factor. Thus we also give the conditions that a product of such Weierstrass \( \sigma \)-functions is a modular function. We shall also see that all these functions essentially generate the same group of units in \( QR_N \), where \( R_N \) is the integral closure of \( \mathbb{Z}[\tau] \) in the modular function field \( F_N \) of level \( N \), over the constant field of \( N \)-th roots of unity.

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§ 1. Recall of Some Formulas

For the convenience of the reader we recall some definitions and formulas, handled in detail in [KL II].

A form of degree \( k \) is a function

\[
  h \left( \frac{\omega_1}{\omega_2} \right), \quad \text{Im} \left( \frac{\omega_1}{\omega_2} \right) > 0,
\]

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Let $F$ be, say, a subgroup of $\text{SL}_2(\mathbb{Z})$. We say that a form as above is modular with respect to $F$ if it satisfies the additional property

$$h\left(\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}\right) = \lambda^k h\left(\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}\right), \quad \lambda \in \mathbb{C}^\ast.$$ 

Let $\eta_1, \eta_2$ be the corresponding quasi-periods of the Weierstrass zeta function associated with the period lattice $[\omega_1, \omega_2]$. Let $N$ be an integer $> 1$, let $r, s$ be integers not both congruent to $0 \mod N$. We define the Klein forms

$$f_{r,s}(\omega_1, \omega_2) = \frac{r}{N}, \frac{s}{N}; \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \exp\left(-\frac{r\eta_1 + s\eta_2}{N} + \frac{r\omega_2 + s\omega_2}{2N}\right)\sigma \left(\frac{r\omega_1 + s\omega_2}{N}; \omega_1, \omega_2\right).$$

If the integer $N$ is fixed throughout a discussion, we index the Klein form $f$ just with $(r, s)$. When we deal with relations between Klein forms of various levels, however, it is necessary to keep the $N$ in the notation. Other possibilities which may also be convenient are

$$f\left(\frac{r\omega_1 + s\omega_2}{N}; \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}\right).$$

Since the sigma function is homogeneous of degree 1, and the quasi periods are homogeneous of degree $-1$, it follows that the Klein form is homogeneous of degree 1 in $[\omega_1, \omega_2]$, that is

$$f_{r,s}(\lambda \omega_1, \lambda \omega_2) = \lambda f_{r,s}(\omega_1, \omega_2), \quad \lambda \in \mathbb{C}^\ast.$$