Semicontinuity of $L$-Dimension

David Lieberman\textsuperscript{1} and Edoardo Sernesi\textsuperscript{2*}

\textsuperscript{1} Department of Mathematics, Brandeis University, Waltham, Mass. 02154, U.S.A.
\textsuperscript{2} Istituto Matematico, Università degli Studi, I-44100 Ferrara, Italia

§ 1. Introduction

Let $X$ be a compact complex space and $L$ and invertible sheaf. The notion of $L$-dimension of $X$, denoted $k(X, L)$, was introduced by Iitaka in \cite{4} (for the definition see § 3). When $X$ is a compact manifold and $L = \omega_X$ the canonical invertible sheaf, $k(X, \omega_X)$ is the canonical dimension of $X$, or the Kodaira dimension of $X$, sometimes denoted $k\dim(X)$; it is the fundamental invariant in the Enriques-Kodaira classification of surfaces.

An important open question is the behaviour of Kodaira dimension under deformation. Not much is known about it. One knows that the canonical dimension is a deformation invariant for curves and surfaces. In the former case the result is trivial, in the latter it has been proved by Iitaka \cite{3} using the classification of surfaces. It is also known that for higher dimensional manifolds the canonical dimension is not a deformation invariant: Nakamura has produced an example of a family of threefolds $\{X_t\}_{t \in \Delta}$ over a disc $\Delta$ such that $k\dim(X_0)=0$ and $k\dim(X_t)=-\infty$ if $t \neq 0$ (cf. \cite{8}).

We investigate the behaviour of $L$-dimension under deformation. Our main results are the following.

**Theorem.** Let $\mathcal{O}$ be an invertible sheaf on a complex space $\mathfrak{X}$ and $\mathfrak{X} \xrightarrow{\pi} S$ a proper and flat morphism onto an irreducible complex space $S$. There is a constant $k$ and a set $W \subseteq S$, which is the complement of the union of a countable number of proper closed subvarieties, such that

- $k(\mathfrak{X}_s, \mathcal{O}_s) = k$ if $s \in W$,
- $k(\mathfrak{X}_s, \mathcal{O}_s) > k$ if $s \in S \setminus W$.

**Theorem.** Let $L$ be an invertible sheaf on a compact, reduced, irreducible complex space $X$ such that the following condition is satisfied for some $n > 0$:

* To whom offprint requests should be sent
if \( k(X, L) \neq -\infty \) there is a polynomial \( Q(T) \) with rational coefficients, of degree less than \( k(X, L) \), such that

\[
\dim c H^1(X, L^{\otimes n}) \leq Q(i) \quad \text{for all } i \gg 0.
\]

Then for every deformation \((X', L')\) of \((X, L)\) (over an irreducible base space) we have \( k(X', L') \geq k(X, L) \).

Condition (a) is satisfied in the important case when \( k(X, L) = \dim X \) and \( L^{\otimes n} \) is spanned by its global sections for some \( n > 0 \) (see Corollary (4.3)). This gives in particular that if \( X \) is a compact manifold of general type and if \( \omega_X^{\otimes n} \) is generated by its global sections for some \( n > 0 \), then every deformation of \( X \) (over an irreducible base space) is again a manifold of general type; this result has also been proved by H. Kurke using different methods. Since Mumford has proved \([5]\) that for surfaces of general type \( \omega_X^{\otimes n} \) is generated by global sections when \( n \gg 0 \), we get as a particular case the result of Iitaka \([3]\) that deformations of surfaces of general type are of general type.

The techniques used are developed in § 2; they are based on strong finiteness theorems (cf. \([2]\) and \([63]\)) and consist of a detailed analysis of the projective data of the problem via graded and homological algebra.

In § 3 we introduce the basic notions; the first theorem is proved and a class of examples is described.

In § 4 we prove the second main result and some corollaries.

§ 2. Preliminaries and Notations

(2.1) Let \( R = \bigoplus_{i \geq 0} R_i \) be a graded \( \mathbb{C} \)-algebra. If \( n \) is a positive integer denote \( R^{(n)} = \bigoplus_{i \geq 0} R_{in} \); then one has a natural identification of schemes \( \text{Proj}(R) = \text{Proj}(R^{(n)}) \).

If \( R = \mathbb{C}[T_0, \ldots, T_N]/H \), where \( H \) is a homogeneous ideal in the polynomial ring \( \mathbb{C}[T_0, \ldots, T_N] \), we say that \( R \) is a homogeneous \( \mathbb{C} \)-algebra; if moreover \( R \) is an integral domain we say that \( R \) is a homogeneous \( \mathbb{C} \)-domain.

If \( R = \bigoplus_{i \geq 0} R_i \) is a graded \( \mathbb{C} \)-algebra and \( n \) is a positive integer we denote \( R^{[n]} = R_0[R_n] \), the graded subalgebra generated by \( R_n \) over \( R_0 \).

(2.2) Lemma. Let \( R = \bigoplus_{i \geq 0} R_i \) be a homogeneous \( \mathbb{C} \)-domain and let \( P(T) \) be its characteristic polynomial \([9]\). Suppose that \( \hat{R} = \bigoplus_{i \geq 0} \hat{R}_i \) is a graded sub \( \mathbb{C} \)-algebra of \( R \) such that \( \hat{R}_i \times R_i \) for all \( i \), satisfying the following condition:

\[\exists \text{ a polynomial } \hat{P}(T) \text{ with rational coefficients, having the same degree and the same leading coefficient as } P(T), \text{ such that } \dim c \hat{R}_i \geq \hat{P}(i) \quad \text{for all } i \gg 0.\]

Then \( R \) and \( \hat{R} \) have the same quotient field.