We consider the estimation of the unknown mean of a homogeneous random field from observations on a system of homothetically expanding regions. We examine the asymptotic behavior of the variance of the arithmetic-mean estimator. The arithmetic-mean estimator is shown to be asymptotically efficient in the class of linear estimators.

Let \( \xi(x_1, \ldots, x_n) \) be a homogeneous random field in the space \( \mathbb{R}^n \). Let \( \xi(x_1, \ldots, x_n) = a + \eta(x_1, \ldots, x_n) \), where \( a \) is an unknown constant and \( \mathbb{E}(\eta(x_1, \ldots, x_n)) = 0 \). We know that \( \eta(x_1, \ldots, x_n) \) has the spectral expansion

\[
\eta(x_1, \ldots, x_n) = \sum_{\lambda \in \Lambda} \sum_{j=1}^n \lambda_j z_j(\lambda)(d\lambda),
\]

where \( z(\cdot) \) is a measure with orthogonal values and \( \mathbb{E}(d\lambda) = 0 \).

Suppose that \( \xi(x_1, \ldots, x_n) \) is observed in the region \( G_T = \{ (x_1, \ldots, x_n): x_j = T t_j, j = 1, \ldots, n, (t_1, \ldots, t_n) \in G \} \), where \( G \) is some region of positive Lebesgue measure \( m(G) \). The region \( G_T \) is obtained from \( G \) by homothetic transformation with the coefficient \( T \).

Consider the arithmetic-mean estimator of the unknown mean \( a \),

\[
\hat{a}_T = \frac{1}{m(G_T)} \int_{G_T} \xi(x_1, \ldots, x_n) \, dx_1 \ldots dx_n.
\]

The estimator \( \hat{a}_T \) is the least-squares estimator. The variance of the arithmetic-mean estimator is

\[
\mathbb{V}(\hat{a}_T) = \frac{1}{m(G_T)} \int_{G_T} \int_{G_T} \int_{G_T} e^{i(x_1, \ldots, x_n)} e^{i(y_1, \ldots, y_n)} \, dx_1 \ldots dx_n dy_1 \ldots dy_n = \frac{1}{m(G_T)} \int_{G_T} \int_{G_T} \int_{G_T} e^{i(x_1, \ldots, x_n)} e^{i(y_1, \ldots, y_n)} \, dx_1 \ldots dx_n dy_1 \ldots dy_n.
\]

where

\[
f(\lambda) = f(\lambda_1, \ldots, \lambda_n), \quad (\lambda, t) = \sum_{j=1}^n \lambda_j t_j, \quad d(\lambda) = d\lambda_1 \ldots d\lambda_n, \quad f\left(\frac{\lambda}{T}\right) = f\left(\frac{\lambda_1}{T}, \ldots, \frac{\lambda_n}{T}\right).
\]

Let

\[
k(u_1, \ldots, u_n) = \int_{G_T} \int_{G_T} \int_{G_T} e^{i(x_1, \ldots, x_n)} e^{i(y_1, \ldots, y_n)} \, dx_1 \ldots dx_n dy_1 \ldots dy_n.
\]

where
The functions $k(\mu_1, \ldots, \mu_n)$ can be computed explicitly for some special regions $G$. For instance, if $G = \{t = (t_1, \ldots, t_n) : -1 \leq t_i \leq 1, i = 1, \ldots, n\}$, then

$$k(\mu_1, \ldots, \mu_n) = 2^n \prod_{k=1}^{n} \frac{\sin \mu_k}{\mu_k};$$

if $G$ is a ball of unit radius, then

$$k(\mu_1, \ldots, \mu_n) = (2\pi)^{n/2} \left( \frac{|\mu|}{|\mu|^n} \right),$$

where $|\mu| = \sqrt{\sum_{k=1}^{n} \mu_k^2}$, $J_0(x)$ is Bessel function of the first kind.

Below we use the following theorem on asymptotic behavior of multiple integrals dependent on a large parameter. **THEOREM 1.** Suppose that in the integral

$$\int_{R^n} \cdots \int S(T\lambda_1, \ldots, T\lambda_n) f(\lambda_1, \ldots, \lambda_n) d\lambda_1 \cdots d\lambda_n,$$

a) the function $f(\lambda_1, \ldots, \lambda_n)$ is bounded on $R^n$ and continuous at the point $(0, \ldots, 0)$;

b) the function $S(\lambda_1, \ldots, \lambda_n)$ is nonnegative and integrable on $R^n$, and

$$\int_{R^n} S(\lambda_1, \ldots, \lambda_n) d\lambda_1 \cdots d\lambda_n = \gamma.$$

Then

$$\int_{R^n} \cdots \int S(T\lambda_1, \ldots, T\lambda_n) f(\lambda_1, \ldots, \lambda_n) d\lambda_1 \cdots d\lambda_n = \gamma f(0) + o(T^{-n}).$$

This assertion is proved similarly to the theorem in [1, p. 30]. We give the proof for $n = 2$. Note that

$$\int_{R^2} S(T\lambda_1, T\lambda_2) d\lambda_1 d\lambda_2 = \frac{\gamma f(0)}{T^2} - \beta_T.$$

Now

$$\int_{R^2} S(T\lambda_1, T\lambda_2) f(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2 = \frac{\gamma f(0)}{T^2} + \int_{R^2} S(T\lambda_1, T\lambda_2) \times$$

$$(f(\lambda_1, \lambda_2) - f(0, 0)) d\lambda_1 d\lambda_2 = \frac{\gamma f(0)}{T^2} + \beta_T.$$

We will show that $\beta_T = o(T^{-2})$.

Let $\alpha_T \to 0$ as $T \to \infty$, but $T\alpha_T \to +\infty$. Then

$$|\beta_T| \leq \sup_{|\mu_1| \leq \alpha_T} |f(\lambda_1, \lambda_2) - f(0, 0)| \int_{R^2} S(T\lambda_1, T\lambda_2) d\lambda_1 d\lambda_2 +$$

$$+ 2 \sup |f(\lambda_1, \lambda_2)| T^{-2} \int_{T\lambda_1 \leq \lambda_1 \leq \infty} S(\lambda_1, u_2) du_1 du_2 + \int_{-\infty < u_1 < \infty} S(u_1, u_2) du_1 du_2.$$

This inequality and the assumptions of the theorem show that $\beta_T = o(T^{-2})$. Q.E.D.

**THEOREM 2.** Assume that the spectral density is continuous at zero and is bounded on $R^n$. Then