It is proved that every symmetric function in $k$-valued logic of $n$ arguments can be realized by a formula in any basis, the complexity of the formula not exceeding $n^C$, where $C$ is a constant depending on the basis. It is shown that in the case $k = 2$, $C \leq 4.93$ for all bases.

Symmetric functions in the algebra of logic were studied by Korobkov [1] and Lupanov [2] from the point of view of the complexity of their realizations by $H$-schemes, or, what is the same thing, by formulae in the basis $\{\lor, \land, \lnot\}$. In the terminology of [2] (we recall that $S_n$ denotes the set of all symmetric functions of $n$ arguments and $L_\pi(\mathcal{E}_n) = \max_{\pi \in \mathcal{E}_n} L_\pi (\pi)$) the fundamental result of Korobkov can be formulated in the form

$$L_n(\mathcal{E}_n) \leq n^{1 + \varepsilon_n} \log n, \quad \varepsilon_n \to 0,$$

where $\varepsilon_n$ is a constant, and it will be shown that $\varepsilon_n \to 0$.

It will be proved in this note that

$$L_n(\mathcal{E}_n) \leq n^c,$$

where $c$ is a constant, and it will be shown that $c \leq 4.93$. Comparing this bound with the lower bound obtained in [3] for the linear function $g_n \in S_n$, we have

$$n^2 \leq L_n(\mathcal{E}_n) \leq n^4 g_n.$$

In view of the results of [4], similar bounds hold for almost all symmetric functions $S_n(x_1, \ldots, x_n)$ depending on $n$ arguments:

$$n^3 \leq L_\pi(S_n) \leq n^{4.05}.$$

For monotonic $S_n^{m_1, \ldots, m_n}(x_1, \ldots, x_n)$ and elementary $S_n^{m_1}(x_1, \ldots, x_n)$ symmetric functions somewhat stronger upper bounds can be obtained:

$$L_n(S_n^{m_1, \ldots, m_n}) \leq n^{4.43},$$

$$L_n(S_n^{m_1}) \leq n^{4.82}.$$
Comparing these with the lower bounds in [4], for \( m \approx \frac{n}{2} \) we have

\[
\begin{align*}
\frac{1}{4} n^2 & \leq L_n(S_n^{m=\ldots=n}) \leq n^{k_{lo}}, \\
\frac{1}{2} n^2 & \leq L_n(S_n^{m=\ldots=n}) \leq n^{k_{hi}}.
\end{align*}
\]

The generalization of (1) presents no difficulties. Let \( \mathcal{S}_n^k \) be the set of all symmetric functions of \( k \)-valued logic of \( n \) arguments and \( L_B(\mathcal{S}_n^k) \) the least number of symbols for variables sufficient to realize any function \( f \in \mathcal{S}_n^k \) by a formula in an arbitrary basis \( B \). Then

\[
L_B(\mathcal{S}_n^k) \leq n^c,
\]

where \( C \) is a constant depending on \( k \) and \( B \), where, for \( k = 2 \), in view of the results of Subbotovskaya (Muchnik) [5], for all bases the constant \( C \) has the same upper bound as in (1). In other words, if \( k = 2 \), \( C \leq 4.93 \).

The bound (1) (without the improvement in the constant \( c \)) is quite simple to deduce from Lupanov's construction [6] for the realization of an arbitrary symmetric function in the algebra of logic of \( n \) arguments by a scheme of functional elements in any finite basis. We only need to verify that such a scheme has depth of order \( \log n \), after which it remains to "develop" the scheme into a formula and estimate the complexity of the formula in terms of the depth.

To prove (4) we have to generalize Lupanov's construction. For arbitrary \( k \) the scheme is constructed from \( k-1 \) blocks computing, respectively, the number of units, pairs, ... and digits, equal to \( k-1 \) in the set of values of the arguments and a further block which produces the value of the function from the results of these computations. This scheme also has depth of order \( \log n \) and so the complexity of the corresponding formula has the upper bound \( n^C \).

We turn to the detailed proof of bounds (1) (with \( c = 4.93 \)), (2), and (3). First we note that, in view of the equation

\[
S_n^m(x_1, \ldots, x_n) = S_n^{m-1}(x_1, \ldots, x_n) \& S_n^{m+1}(x_1, \ldots, x_n)
\]

the bound (3) is a direct corollary of the bound (2). Further, since any symmetric function of \( n \) arguments can be obtained by the substitution of constants from a symmetric function of any greater number of arguments, it is sufficient to prove (1), for example, for all \( n \) of the form \( n = 2^k-1 \), where \( k \) is an integer. Exactly the same holds for monotonic symmetric functions. In addition, any monotonic symmetric function of \( n \) arguments can be obtained by substituting constants from the function \( S_{n+1}^{2^k-1}(x_1, \ldots, x_{2n+1}) \).

Hence, it is sufficient to prove the bound (2) for \( n = 2^k-1 \) and only for \( S_{2n+1}(x_1, \ldots, x_n) \).

Thus, let

\[
n = 2^k - 1.
\]

In the same way as in [6], we can represent an arbitrary symmetric function \( S_n(x_1, \ldots, x_n) \) in the form

\[
S_n(x_1, \ldots, x_n) = f(y_1, \ldots, y_k),
\]

where \( y_1, \ldots, y_k \) are binary digits satisfying the condition:

\[
\sum_{j=1}^{n} x_j = \sum_{i=1}^{k} y_i 2^{k-1}
\]

(both sums are arithmetic!) and \( f \) is a function in the algebra of logic. Clearly, each digit \( y_i = y_i(x_1, \ldots, x_n) \) (1 ≤ \( i \) ≤ \( k \)) is a symmetric function, where

\[
y_i(x_1, \ldots, x_n) = S_{2n+1}^{(n+1)/2, \ldots, n}(x_1, \ldots, x_n).
\]