ON COERCIVE SOLVABILITY OF PARABOLIC EQUATIONS

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In this paper we study the coercive solvability of abstract differential equations of parabolic type in the spaces of L. N. Slobodetskii \( W_p^q \). It is established that the solution of equations with a constant operator \( A \) which generates an analytic semigroup belongs to the trace space \( E(\alpha, p, A) \). The results obtained are applied to the study of equations with a variable operator.

We consider the problem

\[
\frac{dv}{dt} + Av = f(t) \quad (0 \leq t \leq 1), \quad v(0) = v_0
\]

in a Banach space \( E \). In [1, 2] the coercive solvability of this problem is studied in various spaces of functions with values in \( E \). For a broad class of such problems a necessary condition for coercive solvability is the analyticity of the semigroup \( \exp \{-At\} \). Extensive classes of spaces (H"older spaces with a weight, interpolation spaces) have been found for which the analyticity of this semigroup is also a sufficient condition for coercive solvability. In [3] the problem is studied in the spaces \( W_p^q \) of L. N. Slobodetskii. The coercive solvability is deduced from the general theory of the coercive solvability of abstract equations of the form \( Au + Bu = f \). In the present paper a stronger coerciveness inequality is established for solving problem (1). It is also proved that a solution \( v(t) \) of problem (1) for all \( 0 \leq t \leq 1 \) belongs to the trace space \( E(\alpha, p, A) \), if the initial value \( v_0 \) belongs to this space, while the function \( v(t) \) as a function with values in \( E(\alpha, p, A) \) is continuous. This has provided, for example, the possibility of studying the equation with a variable operator and establishing its solvability with fewer restrictions on the smoothness of \( A(t) \) than in [3]. We believe that the simple method of proving coercive solvability of the equation with a constant operator presented in the paper is also of independent interest.

1. For a smooth function \( f(t) \) \( (0 \leq t \leq 1) \) with values in \( E \) we define the norm

\[
\| f \|_{W_p^q} = \int_0^1 \| f(t) \|_E dt + \int_0^1 \int_0^t \frac{| f(t) - f(s) |_E}{| t - s |^{1+\alpha}} d\theta dt.
\]

(2)

The closure of all smooth functions in this norm forms a space \( W_p^q = W_p^q([0, 1], E) \). This space can also be defined directly as the set of strongly measurable functions for which the norm (2) is finite.

**Lemma 1.** For any \( 1 \leq p < \infty \) and \( 0 < \alpha < 1/p \) \( 0 \leq \tau \leq 1 \) \( f(t) \in W_p^q \), we have the inequality

\[
\left( \int_0^1 | t - \tau |^{-\alpha p} \| f(t) \|_E^p dt \right)^{1/p} \leq C(\alpha, p) \| f \|_{W_p^q}.
\]

It is obviously sufficient to carry out the proof for the scalar case. The case \( p = 2 \) has been considered in [4]. The case \( p = 1 \) has been considered in [5]. Another proof is given below.

We suppose first of all that \( \tau = 0 \). We make use of the integral representation of the function \( t^{-\alpha p} \)

\[
t^{-\alpha p} = \frac{1}{\Gamma(\alpha p)} \int_0^1 \lambda^{-\alpha p-1} \exp(-\lambda t) d\lambda + \frac{1}{\Gamma(\alpha p)} \int_0^1 \lambda^{-\alpha p-1} \exp \left\{ -\frac{t}{\lambda} \right\} d\lambda
\]
which follows easily from the formula for the gamma function. From this we have

\[ J = \int_0^\infty t^{-\lambda/p} f(t)^p \, dt = \frac{1}{\Gamma(\alpha p)} \int_0^\infty t^{-\lambda/p} \exp\left\{-\frac{t}{\lambda}\right\} \, dt \]

\[ + \frac{1}{\Gamma(\alpha p)} \int_0^\infty t^{-\lambda/p} \exp\left\{-\frac{t}{\lambda}\right\} \, dx \, dy = J_1 + J_2. \]

(4)

First of all, the following estimate is obvious:

\[ J_1 \leq \frac{1}{\Gamma(\alpha p + 1)} \int_0^\infty t^{-\lambda/p} \, dt. \]

(5)

To estimate the integral \( J_2 \), we interchange the order of integration and decompose the inner integral into a sum of two integrals:

\[ J_2 = \frac{1}{\Gamma(\alpha p)} \int_0^\infty t^{-\lambda/p} \exp\left\{-\frac{t}{\lambda}\right\} \, dt \, dx + \frac{1}{\Gamma(\alpha p)} \int_0^\infty t^{-\lambda/p} \exp\left\{-\frac{t}{\lambda}\right\} \, dx \, dt = J_3 + J_4. \]

Interchanging the order of integration in \( J_1 \) and making a change of variable in the inner integral, we obtain

\[ J_1 = J_1 + J_2. \]

To estimate the integral \( J_2 \), we use the inequality

\[ (a + b)^p \leq (1 + \epsilon) a^p + C_p(\epsilon) b^p, \]

where

\[ C_p(\epsilon) = (1 + \epsilon)((1 + \epsilon) - 1)^{1-p}, \]

which is valid for any positive \( a, b \), and \( \epsilon \). We have

\[ J_2 \leq \frac{C_p(\epsilon)}{\Gamma(\alpha p)} \int_0^\infty \lambda^{-\alpha p-1} \exp\left\{-\frac{t}{\lambda}\right\} \, dt \, dx + \frac{1 + \epsilon}{\Gamma(\alpha p)} \int_0^\infty \lambda^{-\alpha p-1} \exp\left\{-\frac{t}{\lambda}\right\} \, dx \, dt = J_3 + J_4. \]

Computing the inner integral, we find that

\[ J_3 = (1 + \epsilon) (1 - \exp\{-1\}) J. \]

(7)

Finally, from the fact that in the region of integration \( \lambda \geq \lambda - t \), we have the estimate

\[ J_3 \leq \frac{C_p(\epsilon)}{\Gamma(\alpha p)} \int_0^\infty \lambda^{-\alpha p-1} \exp\left\{-\frac{t}{\lambda}\right\} \, dt \, dx. \]

(8)

Since for sufficiently small \( \epsilon \) we have the inequality

\[ (1 + \epsilon)(1 - \exp\{-1\}) + \int_0^\infty \lambda^{-\alpha p-1} \exp\{-\lambda\} \, dx \leq \Gamma(\alpha p), \]

from this and (4)-(8) we obtain the assertion of the lemma for \( \tau = 0 \). In the case of arbitrary \( \tau \) the integral is decomposed into two parts. Large coefficients can hereby occur in estimating the integrals over small segments. In order to get around this, the function \( f(t) \) is first extended (for example, as an even function to a larger interval with preservation of norm) and the result already obtained is applied. This concludes the proof of the lemma.

Suppose that the operator \( A \) is the generator of a strongly continuous semigroup with exponentially decreasing norm. For any \( v_0 \in D(A) \) and \( p \geq 1, \alpha < 1/p \) we set

\[ \| v_0 \|_{E(\alpha, p, A)} = \int_0^1 t^{-\alpha/p} \| A \exp\{-At\} v_0 \|_{L^p} \, dt. \]

(9)

It is not hard to show that the functional \( \| v_0 \|_{E(\alpha, p, A)} \) possesses all the properties of a norm. The closure of \( D(A) \) in this norm forms a Banach space \( E(\alpha, p, A) \). This space can also be defined directly as the set of all \( v_0 \in E \) for which the integral (9) converges. It follows from Lemma 1 that

\[ \| v_0 \|_{E(\alpha, p, A)} \leq C_1(\alpha, p) \| A \exp\{-At\} v_0 \|_{L^p}. \]

Definition. The operator \( A \) is called strongly positive if for the semigroup \( \exp\{-At\} \) the following estimates are valid:

\[ \| \exp\{-At\} \|_E \leq C_1 \exp\{-\delta t\}, \| A \exp\{-At\} \|_{E \to E} \leq C_2 t^{-1} \exp\{-\delta t\}. \]

(10)