NONLINEAR \((n-1)p\)-CONNECTIONS IN METRIC
CARTAN SPACES OF HIGHER ORDER

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Structures of higher order determined on a manifold \(V_n\) by a differential form of degree \(n-1\), which depends on a tangential \((n-1)p\)-element, are considered. The associated nonlinear and linear connections in the corresponding principal fibrations are studied. (See [3] for terminology.)

We agree to call the exterior differential form
\[ \theta = \psi \omega^1 \wedge \omega^2 \wedge \ldots \wedge \omega^{n-1}, \]
invariantly defined on the manifold \(S_{n-1}(V_n)\), a Cartan \(p\)-metric of the manifold \(V_n\).

The form \(\theta\) can also be interpreted in another way. Every submanifold \(V_{n-1} \subset V_n\) generates a holonomic submanifold of tangent \((n-1)p\)-elements \(V_{n-1}^p \subset S_{n-1}^p(V_n)\); then the form \(\theta\) on \(V_{n-1}^p : \theta(V_{n-1}^p)\) may be regarded as a form on \(V_{n-1}\). Thus the possibility arises of the parallel development of two geometries: the internal geometry of the submanifold \(V_{n-1} \subset V_n\) and the geometry of the space \(S_{n-1}^p(V_n)\), furnished with the form \(\theta\). For the construction of the first geometry we use the concept of a nonlinear \(mp\)-connection developed in [1]. For the construction of the geometry of \(S_{n-1}^p(V_n)\) we find the corresponding linear connection in some fibration for which \(S_{n-1}^p(V_n)\) also serves as a basis.

The first attempt at a study of metrics of higher order \((n = 2, p = 2)\) is found in [2]. Thereafter this investigation was continued in [3] (with arbitrary \(n, p = 2\)). In the present paper the case where both \(n\) and \(p\) are arbitrary is considered.

A relative invariant \(r\) is attached to each point of the space \(S_{n-1}^p(V_n)\) with the canonical forms \(\omega^k, \omega^a, \omega^b, \ldots, \omega^{a_1 \ldots a_p}\) \((a, b, c = 1, 2, \ldots, n-1)\) and our problem starts with the equation
\[ d\varphi - \varphi \omega^a = \varphi_{a} \omega^a + \varphi^{a} \omega^a + \ldots + \varphi^{a_1 \ldots a_p} \omega^{a_1 \ldots a_p}. \tag{1} \]
The basis forms \(\omega^k, \omega^a, \ldots, \omega^{a_1 \ldots a_p}\) will occasionally be written \(\omega^R\), for short, where \(R, S, Q, \ldots\) are corresponding collective indices. Because of the symmetry of the forms \(\omega^{a_1 \ldots a_p}\) the coefficients \(\varphi^{a_1 \ldots a_p}\) are also symmetric in the indices \(a_1 \ldots a_p\). Taking account of the structure equations of the basis forms
\[ \omega^{a_1 \ldots a_p} = \sum_{s=0}^{r} \omega^{a_1 \ldots a_s} \wedge \omega^{b_1 \ldots b_t} \wedge \omega^{c_1 \ldots c_t}, \tag{2} \]
\[ \omega^{a_1 \ldots a_p} = \frac{r!}{(r-s)!} \delta_{a_1 \ldots a_s}^{b_1 \ldots b_t} \omega^{b_1 \ldots b_t} \wedge \omega^{c_1 \ldots c_t}, \tag{3} \]
we find a series of extensions of the original Eq. (1). The first extension of (1) gives the equations
\[ d\varphi - \varphi_{a} \omega^a - \varphi^{a} \omega^a - \varphi^{a_1 \ldots a_p} = \varphi_{a_1 \ldots a_p} = \varphi_{a_1 \ldots a_p} = \varphi_{a_1 \ldots a_p}, \tag{4a} \]


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\[ d\alpha_n - q^2 \omega^b_n - \omega^a_n = \sum_{\beta=1}^{p} q^{b_i \ldots b_p} \omega^{a \ldots a}_{n \beta} = \varphi_{n R} \omega^R, \]  
\[ d\psi^{(b_1 \ldots b_p)}_n - \sum_{\beta=1}^{p} q^{b_1 \ldots b_p} \omega^{a \ldots a}_{n \beta} = \varphi_{n R} \omega^R \]  
\( r = 2, \ldots, p - 1, \)
\[ d\omega^{a \ldots a}_{b_1 \ldots b_p} - q^{a \ldots a}_n \omega^b_n + \sum_{\beta=1}^{p} q^{b_i \ldots b_p} \omega^{a \ldots a}_{n \beta} = \varphi_{n R} \omega^R + \sum_{\beta=1}^{p} q^{a \ldots a}_n b_i \cdot b_p \omega^{a \ldots a}_{b_i \ldots b_p}, \]  
(4c)

where \( \varphi_{RS} = \varphi_{SR} \) by Cartan's lemma.

Then we write down the extension of Eq. (4c) 
\[ d\psi^{(b_1 \ldots b_p)}_n + \{ (b_1 \ldots b_p) \} - q^{a \ldots a}_n \omega^b_n \] 
\[ \omega^{a \ldots a}_{n \beta} - \sum_{\beta=1}^{p} q^{b_i \ldots b_p} \omega^{a \ldots a}_{n \beta} = \varphi_{n R} \omega^R. \]  
(5a)

(The symbol \( \{ (b_1 \ldots b_p) \} \) is an abbreviated notation for the tensorial linear combination of forms \( \omega^b_n \); \( \{ a \ldots a \} \) denotes the same combination, but for the forms \( \omega^a_n \).)

\[ d\psi^{(b_1 \ldots b_p)}_n + \{ (b_1 \ldots b_p) \} - q^{a \ldots a}_n \omega^b_n + \omega^{a \ldots a}_{n \beta} - \sum_{\beta=1}^{p} q^{b_i \ldots b_p} \omega^{a \ldots a}_{n \beta} = \varphi_{n R} \omega^R, \]  
\[ d\psi^{(b_1 \ldots b_p)}_n + \{ (b_1 \ldots b_p) \} - q^{a \ldots a}_n \omega^b_n - \sum_{\beta=1}^{p} q^{b_i \ldots b_p} \omega^{a \ldots a}_{n \beta} = \varphi_{n R} \omega^R, \]  
\[ d\omega^{a \ldots a}_{b_1 \ldots b_p} + \{ (b_1 \ldots b_p) \} - q^{a \ldots a}_n \omega^b_n + \omega^{a \ldots a}_{n \beta} - \sum_{\beta=1}^{p} q^{b_i \ldots b_p} \omega^{a \ldots a}_{n \beta} = \varphi_{n R} \omega^R. \]  
(5b)

The last group of equations can be described in a way analogous to the description of Eq. (4c); the quantities \( q^{a \ldots a}_n b_i \cdot b_p \) remain unchanged upon permutation of the indices \( a_1 \ldots a_p \), and the indices \( b_1 \ldots b_p \), and also upon interchange of the sets of indices \( (a_1 \ldots a_p) \) and \( (b_1 \ldots b_p) \). Nevertheless, in the general case one can introduce a system of quantities \( q^{a \ldots a}_n b_i \cdot b_p \) in a unique way whose indices have the same property where their indices are concerned and such that
\[ q^{a \ldots a}_n b_i \cdot b_p = \delta^{a_i \ldots a_p}_n b_i \cdot b_p. \]  
(6)

This enables us to introduce the following system of quantities:
\[ q^{a \ldots a}_n b_i \cdot b_p = \delta^{a_i \ldots a_p}_n b_i \cdot b_p. \]  
(7)

Without immediately writing down the equations satisfied by the \( q^{a \ldots a}_n b_i \cdot b_p \), we only remark that of the forms (3), those with the greatest number of indices enter these equations as the following type of contraction:
\[ \omega^{a \ldots a}_n b_i \cdot b_p = \delta^{a_i \ldots a_p}_n b_i \cdot b_p, \]  
(3)

Further extension of the original equation will be combined with the minimum amount of canonization necessary to obtain forms characterizing the connection we are seeking.

The first step of the canonization,
\[ q^{a \ldots a}_n b_i \cdot b_p = 0, \]  
(8)
leads immediately to the following equations:
\[ C q^{a \ldots a}_n b_i \cdot b_p = q^{a \ldots a}_n b_i \cdot b_p, \]
\[ - q^{a \ldots a}_n b_i \cdot b_p = q^{a \ldots a}_n b_i \cdot b_p, \]
\[ \omega^{a \ldots a}_n b_i \cdot b_p = q^{a \ldots a}_n b_i \cdot b_p \]
\[ \omega^{a \ldots a}_n b_i \cdot b_p = q^{a \ldots a}_n b_i \cdot b_p. \]  
(9)