Variations on a Theme by Mikhlin

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Introduction

In [5] MIKHLIN develops the $L^2$ theory of singular integral operators on a simple closed plane curve $\Gamma$ of class $C^3$. His main results are:

(a) The operator $H$, defined by

$$ (H \varphi)(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\varphi(\zeta)}{\zeta - z} d\zeta, \quad z \in \Gamma $$

is a linear bounded operator from $L^2(F)$ into $L^2(F)$.

(b) If $p$ and $q$ are continuous functions defined on $\Gamma$, then the operator $T$, defined by

$$ T \varphi = p \varphi + q H \varphi + K \varphi $$

where $K$ is an arbitrary compact operator in $L^2(F)$, admits a regularization and its range is closed. If, further, $p^2(z) - q^2(z) \equiv 0$ for $z \in \Gamma$, then its index is finite.

Later, SEELEY in [7] develops the $L^p(1 < p < \infty)$ theory of singular integral operators on compact manifolds of any dimension and there the notion of symbol of a singular integral operator plays a central role as it does in the theory of CALDERÓN and ZYGMUND.

In this paper we consider in a $L^p(1 < p < \infty)$ setting the singular integral operator $T$ and generalize to the $L^p$ spaces the results (a) and (b) of MIKHLIN. The symbol of $T$ is computed, and it is shown that the condition $p^2(z) - q^2(z) \equiv 0$ for $z \in \Gamma$ is equivalent to the condition that the symbol of $T$ never vanishes, i.e., $T$ is an elliptic operator. (Singular integral operators with non-vanishing symbol nowadays are called elliptic operators.)

FICHERA in [1] and KHVEDELIZE in [4] also consider the $L^p(1 < p < \infty)$ theory of the singular integral operator $T$; however, their approach and tools are different from ours. The central idea in our presentation is to use general results obtained by SEELEY for singular integral operators on compact manifolds in order to generalize the result (b) of MIKHLIN to the $L^p$ case. Our proof of the boundedness in $L^p(\Gamma)$ of the operator $H$ is based upon a theorem of MARCEL RIESZ [6].
§ 1

Let \( \Gamma \) denote a simple closed plane curve \( z = \gamma(s), 0 \leq s \leq L \), where \( \gamma \in C^2[0, L] \). Taking as parameter \( s \) the arc length one obtains \( |\gamma'(s)| = 1 \), \( 0 \leq s \leq L \). (\( \gamma \) stands for \( \frac{d\gamma}{ds} = \frac{dx}{ds} + i\frac{dy}{ds} \)).

Let \( L^p(\Gamma), 1 < p < \infty \), denote the Banach space of the (classes of) functions \( \varphi \), defined on \( \Gamma \), such that the composite function \( \varphi \cdot \gamma \in L^p[0, L] \). We shall consider here the integral

\[
\int_{\Gamma} \frac{\varphi(\zeta)}{\zeta - z} \, d\zeta, \quad z \in \Gamma
\]

as a Cauchy's principal value for functions \( \varphi \in L^p(\Gamma) \), so that for \( z = \gamma(t), 0 < t < L \), one has

\[
\int_{\Gamma} \frac{\varphi(\zeta)}{\zeta - z} \, d\zeta = \lim_{\epsilon \to 0} \left\{ \int_{0}^{t-\epsilon} \frac{\varphi(\gamma(s))}{\gamma(s) - \gamma(t)} \, ds + \int_{t+\epsilon}^{L} \frac{\varphi(\gamma(s))}{\gamma(s) - \gamma(t)} \, ds \right\}.
\]

We will write this relation in the form

\[
\int_{\Gamma} \frac{\varphi(\zeta)}{\zeta - z} \, d\zeta = \int_{0}^{L} \varphi(\gamma(s)) \frac{\dot{\gamma}(s)}{\gamma(s) - \gamma(t)} \, ds.
\]

It is well known that when \( \varphi \) is a Hölder continuous function on \( \Gamma \) then the function \( \psi \):

\[
\psi(z) = \frac{1}{\pi i} \int_{\Gamma} \frac{\varphi(\zeta)}{\zeta - z} \, d\zeta
\]

is defined everywhere and is a Hölder continuous function on \( \Gamma \). When \( \varphi \in L^p(\Gamma) \), then the situation is described by

**Theorem 1.** If \( \varphi \in L^p(\Gamma) \), then the function

\[
\psi(z) = \frac{1}{\pi i} \int_{\Gamma} \frac{\varphi(\zeta)}{\zeta - z} \, d\zeta
\]

is defined almost everywhere on \( \Gamma \) and the operator \( H, H \varphi = \psi \), is a linear bounded operator from \( L^p(\Gamma) \) into \( L^p(\Gamma) \).

**Proof:** First, we write \( \Gamma \) in the form \( z = \gamma_1(s), 0 \leq s \leq 2\pi \), with \( \gamma_1 \in C^2[0, 2\pi], |\gamma_1(s)| \neq 0 \). This can be achieved by means of a homothetic transformation of the parameter, namely, by setting

\[
\gamma_1(s) = \gamma\left(s - \frac{L}{2\pi}\right).
\]

One has to show that the inequality

\[
(1) \quad \int_{0}^{L} |(H \varphi)(\gamma_1(t))|^p \, dt \leq C \int_{0}^{L} |\varphi(\gamma(s))|^p \, ds
\]

holds, which is equivalent to

\[
(2) \quad \int_{0}^{2\pi} |(H \varphi)(\gamma_1(t))|^p \, dt \leq C \int_{0}^{2\pi} |\varphi(\gamma_1(s))|^p \, ds.
\]