BACKLUND TRANSFORMATION FOR INTEGRABLE SYSTEMS

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We establish an explicit form of the Backlund transformation for the best known integrable systems.

1. In this paper the Backlund transformation and its integration for the best known and applicable integrable systems are obtained. Here, by the Backlund transformation we mean any nonlinear mapping which transfers any given solution into another one. However, in this we do not investigate the properties of the transformation, its geometric interpretation (if any), etc. At present we do not have a method for a construction of the transformation in question. To prove its validity, one can make a direct check which uses only one operation — differentiation. As a hint for obtaining the Backlund transformation, the author used a purely algebraic method for construction of the soliton-type solutions (see [1, 2]), modified for the case of the solvable algebras [3].

2. The starting point of our construction below uses the following two facts. The integrable systems under consideration admit the transformation $s$:

\[
\theta \Rightarrow \tilde{\theta} \equiv s\theta = F(\theta, \theta_1, \ldots, \theta_N), \quad S^N \neq 1.
\]

Here $\theta$ and $\tilde{\theta}$ are unknown functions (variables) satisfying the corresponding PDEs, $\alpha_i = \partial^i \theta / \partial x_i$. There is the obvious solution of the nonlinear system in question which depends on a set of arbitrary functions. The soliton-type solutions, reductions related to the discrete groups and solutions with definite boundary conditions, are defined by the special choice of arbitrary functions mentioned above. Let us note that $\theta_0$ is a solution of a linear system of partial differential equations that can be presented as a parametric integral on the plane of the complex variable $\lambda$. This circumstance is just the main reason for applying to the integrable systems the methods of the theory of functions of complex variables, the techniques of the Riemann problem, and, finally, the methods of the inverse scattering problem. The results of the present paper reduce the inverse scattering method to a simple technical rule.

3. Here we give a list of integrable systems together with their Backlund transformations and the corresponding solutions.

1. Hirota equation

\[
\begin{align*}
v' + \alpha(v - 6uv) - i\beta(v - 2v^2) + \gamma \hat{v} + i\delta v &= 0, \\
u' + \alpha(u - 6uv) + i\beta(u - 2u^2) + \gamma \hat{u} - i\delta u &= 0;
\end{align*}
\]

\[
\theta' \equiv \frac{\partial}{\partial t}, \quad \equiv \frac{\partial}{\partial z};
\]

\[
\hat{v} \equiv s\hat{v} = \frac{1}{u}, \quad \hat{u} \equiv su = u(\nu - 1), \quad \nu_0 = 0,
\]

\[
v'_0 + \alpha \hat{u}_0 + i\beta \hat{u}_0 + \gamma \nu_0 - i\delta \nu_0 = 0.
\]

In this and other cases the main role will be played by the principal minors of the following matrix:

\[
\begin{pmatrix}
\phi & \phi + 1 & \phi + 2 & \cdots \\
\phi + 1 & \phi + 2 & \phi + 3 & \cdots \\
\phi + 2 & \phi + 3 & \phi + 4 & \cdots \\
\cdots & \cdots & \cdots & \cdots
\end{pmatrix}
\]
To denote the principal minors of these matrices, we shall use the symbol $D^n_r$. Here $n$ is the rank of the matrix and $r$ is the symbol of its element of the left upper corner. For the solution of the Backlund transformation, we have

$$
v_n = (-1)^n \frac{D^{n-1}_0}{D^0_0}, \quad u_n = (-1)^{n+1} \frac{D^{n+1}_0}{D^0_0},
$$

The methods of the theory of the function of complex variables give the same expression where the nonlocal integral plays the role of $D^n_0$:

$$D^n_0 = \int d\lambda_1 \ldots d\lambda_n c(\lambda_1) \ldots c(\lambda_n) W_n^2(\lambda_1, \ldots, \lambda_n),$$

where $W_n(\lambda)$ is the Vandermonde determinant and $c(\lambda)$ is the integrand in the representation for $u_0$.

2. Nonlinear Schrödinger equation

a) 

$$q' + \bar{q} - 2rq^2 = 0 \quad \bar{q} = \frac{1}{\bar{r}} \quad \bar{r} = r[rq - \ln r];$$

$$-r' + \bar{r} - 2qr^2 = 0 \quad q_0 = 0 \quad r'_0 = \bar{r}_0.$$

The solution of the Backlund transformation is the same as in the previous section.

b) 

$$q' + \bar{q} + 2(rq)q = 0 \quad \bar{q} = \frac{1}{\bar{r}} \quad \bar{r} = r[(rq) + \ln r];$$

$$-r' + \bar{r} - 2(rq)\bar{r} = 0 \quad q_0 = 0 \quad r'_0 = \bar{r}_0.$$

The solution of the Backlund transformation is as follows:

$$q_n = (-1)^n \frac{D^{n-1}_1}{D^0_0}, \quad r_n = (-1)^{n+1} \frac{D^{n+1}_0}{D^0_1},$$

3. One-dimensional Heisenberg ferromagnetic in the classical region (XXX-model).

$$S' = [S, \tilde{S}], \quad S = (S_-, S_0, S_+), \quad S_0^2 + S_+ S_- = 1;$$