maps the circle $\{z: |z-e_j|<\varepsilon\}$ onto a simply connected domain in $\mathbb{C}_\varepsilon$, containing the origin; moreover, the arcs of the trajectories of the differential $Q(z)\,dz$ are mapped into radius segments emanating from the point $w = 0$. This proves that in the neighborhoods of the points $e_j$, $Q(z)$ has the expansion

$$Q(z) = a_j^2/4\pi^2(z-e_j)^2 + \ldots.$$ 

The statements regarding the lengths of the corresponding trajectories and of the arcs of the orthogonal trajectories follow from the definition of $Q(z)\,dz$. This proves Theorem 1.

**Remark 2.** As it follows from Theorem 1, the collection of the inequalities (2.2) for all possible $a$ and $b$, together with the inequalities $x_i \geq 0$, $1 \leq i \leq p$, describes the convex body

$$\mathbb{W} = \left\{ (x_1, \ldots, x_{p+q+1}) : \begin{align*}
x_i &= M(\widehat{b}_j), \quad 1 \leq i \leq p; \\
x_{p+j} &= M(\hat{b}_{p+j}), \\
x_{p+q+k} &= M(\hat{b}_{p+q+k}), \quad 1 \leq j \leq q; \\
x_{p+q+k} &= M(\hat{b}_{p+q+k}), \quad 1 \leq k \leq q.
\end{align*} \right\}$$

of the values of the moduli and reduced moduli for all systems $\hat{b}$ of nonoverlapping domains in $\mathbb{C}$, satisfying the conditions of this theorem, in the same way as the collection of the inequalities (3.11), together with the inequalities $x_i \neq 0$, $1 \leq i \leq p$, describes the body $V$.

**LITERATURE CITED**


**EXPLICIT PRECONDITIONING OF SYSTEMS OF LINEAR ALGEBRAIC EQUATIONS**

**WITH DENSE MATRICES**

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For the acceleration of the convergence of the iterative methods for solving systems of linear algebraic equations with dense matrices, one suggests the use of sparse explicit preconditioners, based on the minimization of quadratic functionals and admitting adaptive refinement. One gives the results of test computations for exterior potential flow problems.

1. **Introduction**

At the solution of a series of computational problems there arises the necessity of solving systems of linear algebraic equations (SLAE) with dense, generally nonsymmetric matrices. For example, such matrices appear at the use of the capacity matrix technique for solving linear elliptic partial differential equations with separable variables and at the use of the method of boundary integral equations; moreover, the dimension of the occurring SLAE, especially in the three-dimensional case, is very large: thousands and tens of thousands. The asymmetry of the coefficient matrix and the absence of its diagonal dominance

lead to the fact that the use of direct methods for solving such systems, based on triangular factorization, becomes practically unacceptable, basically because of the necessity for the search of a pivot element at the factorization stage from the entire dense matrix with the purpose of ensuring the numerical stability of the solving process, which requires an extremely large cost in time, and, in particular, because of the necessity of transmitting information constantly from slow exterior memory devices to the fast operative memory. In this situation, the use of iterative methods for solving SLAE is more perspective or even the only possible one. However, the density and dimension of the coefficient matrix lead to the fact that the cost of each iteration is extremely large. Therefore, the problem of the acceleration of the convergence of iterative methods for systems with dense matrices acquires a fundamental significance. In the considered situation, most natural is the use of the preconditioning of the initial SLAE $Ax = b$, i.e., switching to an equivalent system of the form $GAH y = Gb$, $Hy = x$, where $G$ and $H$ are some nonsingular matrices such that the matrix $GAH$ of the new system is in some sense closer to the identity matrix than the initial one. However, the preconditioning methods, applied successfully for the solving of SLAE with sparse coefficient matrices, such as the incomplete factorizations, polynomial preconditionings, and methods based on the splitting of the initial matrix, become unsuitable in the considered case. This is connected with the fact that the density of the initial matrix and the absence of good functional properties for it do not allow the use of traditional methods for selecting the structures of the triangular factors of the incomplete factorization and the form of the splitting, and also there is an unacceptable increase of the iteration cost for the preconditioned system if the preconditioner matrix is not sufficiently sparse. Thus, the block diagonal scaling turns out to be essentially the unique traditional preconditioning method, potentially applicable in the considered situation. However, one cannot count on its efficiency if elements, sufficiently large in absolute value, are scattered in it chaotically; this is confirmed by the results of numerical experiments (see Sec. 5). An attempt to use the sparse preconditioning of a symmetric dense matrix with diagonal dominance, arising at the use of the capacity matrix technique in the two-dimensional case, is summarized in [1], the utmost conciseness of which does not give a clear image of the presented strategy.

In the present paper, for the acceleration of the convergence of the iterative methods of solving SLAE with dense nonsymmetric matrices, we suggest the use of sparse explicit preconditioners, based on the minimization of quadratic functionals. The application of this approach to systems with sparse symmetric matrices has been considered in [2, 3]. We note that since, for a sufficiently sparse preconditioner, the cost of the preconditioning is small in comparison with that of the multiplication of a dense matrix by a vector, even a reduction of the number of iterations by 10-15% turns out to be advisable from a practical point of view. Another merit of the suggested preconditionings is the possibility of its efficient implementation on parallel computing systems.

The theoretical foundations of the suggested method are presented in Sec. 2. Strategies for the selection of the structure of the preconditioner are considered in Sec. 3. A brief description of a demonstration problem is given in Sec. 4. Section 5 contains the results of numerical experiments.

2. Construction of Explicit Sparse Preconditioners

Assume that one has to solve the SLAE

$$Ax = b$$

(1)

with a square nonsingular matrix $A = (a_{ij})$ of order $n$. Our problem consists in the construction of a sparse nonsingular preconditioner matrix $G$, approximating the inverse matrix $A^{-1}$, with the purpose of switching to an equivalent preconditioned system

$$GAx = Gb$$

(2)

for ensuring a faster convergence of the iterative methods of solution. In this paper we consider the method Orthomin $(k)$ [4, 5], which is one of the generalizations of the conjugate residual method to nonsymmetric positive definite matrices. We recall that a matrix $B$ is said to be positive definite if all the eigenvalues of its Hermitian part $\frac{1}{2}(B^*B)$ are positive.

For the construction of the preconditioner $G$ we make use of an approach based on the minimization of quadratic functionals [2, 3]. We fix some sparsity structure for the matrix $G = (g_{ij})$, i.e., the set $S$ of those positions $(i,j)$, $1 \leq i, j \leq n$, $i \neq j$, off the main diagonal, at