
EXTREMAL PROPERTIES OF QUADRATIC DIFFERENTIALS WITH STRIP-SHAPED DOMAINS IN THE STRUCTURE OF THE TRAJECTORIES

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One considers the modulus problem for a family $\mathcal{H}$ of homotopic classes $\{H_i\}$, in the extended complex plane $\hat{C}$, of the following types. The classes $H_i$ consist of closed Jordan curves, homotopic to appropriate nondegenerate contours or point curves, and also of arcs with endpoints in (distinct or coinciding) distinguished points in $\hat{C}$. One establishes the relation of the indicated extremal metric problem and the problem on the extremal decomposition of $\hat{C}$ in the family $\mathcal{D} = \mathcal{D}_\mathcal{H}$ of systems of mutually nonoverlapping domains $\{D_i\}$, associated with the family $\mathcal{H}$ of the classes $\{H_i\}$. The results of this paper complement a previous theorem of the author [Moduli of families of curves and quadratic differentials. Trudy Mat. Inst. Akad. Nauk SSSR, Vol. 139, 1980].

INTRODUCTION

1°. A series of investigations of a general character, which have started with Jenkins' paper [1], have been devoted to the extremal properties of quadratic differentials with closed trajectories; regarding these questions, see, for example, [2] and the recent monograph by Strebel [3]. In particular, one has shown [2, Chap. 0] that the extremal metric problem $P_\mathcal{H} = P_\mathcal{H}(a_1, \ldots, a_{j+m})$ for the family $\mathcal{H}$ of homotopic classes $\{H_i\}_{i=1}^{j+m}$ of closed curves in the extended plane $\hat{C}$, where $H_i$, $i = 1, \ldots, j$, are classes of curves homotopic to nondegenerate contours, $H_{j+l}$, $l = 1, \ldots, m$, are classes of curves homotopic to point curves in the distinguished points on $\hat{C}$, is directly connected with the so-called problem on the extremal decomposition of $\hat{C}$ into the family $\mathcal{D}_\mathcal{H}$ of all systems of nonoverlapping domains $\{D_i\}_{i=1}^{j+m}$, associated with family $\mathcal{H}$ of classes $H_i$. By the latter we mean the problem of the maximum of the functional

$$
\sum_{i=1}^j \sigma_i^2 \mathcal{M}(D_i) + \sum_{l=1}^m \sigma_{j+l}^2 \mathcal{M}(D_{j+l}, b_l)
$$

in the family $\mathcal{D}_\mathcal{H}$; here $\mathcal{M}(D_i)$ is the modulus of the doubly connected domain $D_i$, associated with the class $H_i$ (i.e., the modulus of the domain $D_i$ relative to the family of curves separating its boundary components), $\mathcal{M}(D_{j+l}, b_l)$ is the reduced modulus of the simply connected domain $D_{j+l}$ relative to the point $b_l \in D_{j+l}$, $\sigma_i$ and $\sigma_{j+l}$ are the positive parameters occurring in the definition of the modulus problem $P_\mathcal{H}$. The relationship between the modulus problem $P_\mathcal{H}$ and the indicated extremal problem in the family $\mathcal{D}_\mathcal{H}$ consists in the following. The extremal metric of the modulus problem $P_\mathcal{H}$ is the $Q$-metric $|Q(z)|^2|dz|$, where $Q(z)dz^2$ is a quadratic differential, regular on $\hat{C}$, with the exclusion of the poles in the distinguished points in $\hat{C}$, and the critical trajectories of this same differential determine the extremal decomposition of $\hat{C}$: the system $\{D_i^*\}_{i=1}^{j+m}$ of annular and circular domains of the differential $Q(z)dz^2$ yields the maximum of the sum (1) in the family $\mathcal{D}_\mathcal{H}$.  

2°. In this paper, the mentioned general principle is extended to the case when the family $\mathcal{H}$ contains the homotopic classes $H_{j+m+1}, \ldots, p$, of arcs, having limiting endpoints

at the distinguished point in \( \mathbb{C} \). The domains associated with the classes of such arcs are the biangles, i.e., simply connected domains with two distinguished boundary elements. With such an extension of the family \( \mathcal{H} \), new aspects arise. Thus, in a number of parameters, defining the modulus problem \( \mathcal{P}_\mathcal{H} \) in this case and the related problem of extremal decomposition, there occurs a system of real numbers \( \{ \gamma_s \}^p_{s=1} \), which are lower bounds for the so-called reduced arc lengths from the classes \( H_{j+m+s} \) in the admissible \( p \)-metric. As shown by simple considerations, in contrast with other parameters, the system \( \{ \gamma_s \} \) cannot be selected in a completely arbitrary manner. However, for all those systems of values of the parameters for which there exists an associated quadratic differential \( Q(z)dz^2 \), there exists a simple relation between the extremal metric problem and the problem of the extremal decomposition: the metric \( |Q(z)|^2dz \) is extremal in the modulus problem \( \mathcal{P}_\mathcal{H} \) and the system \( \{ \gamma_s \}^p_{s=1} \) of annular, circular, and strip domains of the differential \( (2z)^n \) has the following extremal property.

In the admissible family \( \mathcal{D}_\mathcal{H} \) of systems of domains \( \{ \mathcal{D}_i \}_{i=1}^1 \), associated with the family \( \mathcal{H} = \{ H_{j+m+s} \}^p_{j=1} \), we have the inequality

\[
\sum_{i=1}^1 \alpha_i^2 [\mathcal{M}(\mathcal{D}_{i}) - \mathcal{M}(\mathcal{D}_j)] + \sum_{k=1}^m \alpha_{j+k}^2 [\mathcal{M}(\mathcal{D}_{j+k}) - \mathcal{M}(\mathcal{D}_{i+k})] + \\
+ \sum_{s=1}^p \gamma_s^2 [\mathcal{M}(\mathcal{D}_{j+m+s}) - \mathcal{M}(\mathcal{D}_{i+m+s})] \frac{\mathcal{M}(\mathcal{D}_{j+m+s})}{\mathcal{M}(\mathcal{D}_{i+m+s})} > 0 ;
\]

here \( \mathcal{M}(\mathcal{D}_{j+m+s}) \) and \( \mathcal{M}(\mathcal{D}_{i+m+s}) \) are the reduced moduli of the biangles \( \mathcal{D}_{j+m+s} \) and \( \mathcal{D}_{i+m+s} \) with respect to their distinguished boundary elements.

We mention that some of the considered questions are directly related with Emel'yanov's paper [4], which will be mentioned again in the sequel.

Section 1 of the present paper is devoted to the definitions of the reduced modulus of a biangle relative to the corresponding classes of curves and to the extremal metric questions related with these definitions.

Section 2 is devoted to the proof of the fundamental result of the paper.

1. Definition of the Reduced Modulus of a Biangle and Related Extremal Metric Problems

1°. Let \( D \) be a simply connected domain in \( \mathbb{C} \) with two distinguished boundary elements, whose supports are the distinct or the coinciding points \( b_1 \) and \( b_2 \). For the sake of definiteness, we shall assume that \( b_1, b_2 \) are finite; the definitions given below can be carried over in an obvious manner, for example, to the case \( b_1 = \infty \). We shall assume that in the neighborhoods of the points \( b_k, k = 1, 2 \), the boundary arcs of the domain \( D \) behave similarly to the trajectories of the quadratic differential with expansion

\[
Q(z)dz^2 = A(z-b_k)^2(I + O(z-b_k))dz^2, \quad A_k > 0. \tag{2}
\]

Namely, assume that the following condition (\( \mathcal{A} \)) holds: for the mapping \( w = g(z) \) of the domain \( D \) onto the strip \( \Pi = \{ u : h_u < u < h_u \} \), normalized by the conditions \( \text{Re} g(b_k) = -\infty, \text{Re} g(b_2) = +\infty \), in the neighborhoods of the points \( b_k, k = 1, 2 \), we have the equality

\[
q(z) = \sqrt{A_k} \{ (-1)^{k-1} \log(z-b_k) + \bar{q}_k(z) \}, \tag{3}
\]

where \( \bar{q}_k(z) \) is a regular function.

Let \( \varepsilon_k, k = 1, 2 \), be sufficiently small positive numbers. Then the intersection of the domain \( D \) with the circumference \( C(b_k, \varepsilon_k) = \{ z : |z-b_k| = \varepsilon_k \} \) consists of a unique arc, denoted by \( S_k(\varepsilon_k) \). Let \( K(b_k, \varepsilon_k) = \{ z : |z-b_k| < \varepsilon_k \} \), let \( K(b_k, \varepsilon_k) \) be the closure of \( K(b_k, \varepsilon_k) \), and let \( D(\varepsilon_k, \varepsilon_k) = D \setminus K(b_k, \varepsilon_k) \). Thus, \( D(\varepsilon_k, \varepsilon_k) \) is a quadrangle with opposite sides \( S_k(\varepsilon_k), k = 1, 2 \). Let \( \gamma_k(\varepsilon_k) \) be the