GENERALIZED QUADRANGLES, SYMMETRIZATION, AND NONUNIVALENT MAPPINGS

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In this paper one introduces R- and S-transformations of admissible metrics and for 0-univalent mappings one proves theorems on the variation of the moduli of generalized quadrangles under these transformations.

In this paper we introduce the R- and S-transformations of admissible metrics and we prove theorems on the variation of the moduli of the corresponding families of curves under these transformations.

1. Families $\Gamma_1$, $\Gamma_2$ and Their Moduli

Let $L$ be a closed, piecewise analytic curve in the extended complex plane $\mathbb{C}$, separating it into two domains $G_1$ and $G_2$, $0 \in G_2$. Let $b_1$ be a closed Jordan analytic arc of the curve $L$, let $l_1$ be a closed analytic arc (self-intersections allowed), joining some fixed point $c_1$ of the domain $G_1$ with an interior point $c_2$ of the arc $b_1$. Moreover, we require that the arc $l_1$ lie entirely in $G_1$, except its endpoint $c_2$, dividing $b_1$ into two arcs $b_{11}$, $b_{12}$.

Let $D$ be some fixed domain of the plane $\mathbb{C}$, $\mathbb{C} \cup \mathbb{C} \cup \{0\} \subset D$. Then $D_\gamma = (G \cap D) \setminus \ell_\gamma$ is an open set in $\mathbb{C}$. By $\Gamma_\gamma = \Gamma(\ell_\gamma, \ell_\gamma, D_\gamma)$ we denote the family of curves in $D_\gamma$, separating in $D_\gamma$ the curve $\ell_\gamma$ from $D_\gamma \setminus \mathbb{C}$. Unless otherwise specified, here and in the sequel, by a curve we mean a finite union of closed arcs or closed curves, each of which is locally rectifiable.

We consider the continuum $\Omega$, $\emptyset \subset \Omega$, $\Omega \subset D$, the boundary $\partial \Omega$ of which consists of Jordan closed analytic arcs $b_{21}$, $b_{22}$, $b_1$ and a closed analytic arc $l_2$ (self-intersections are allowed). Here we require that the following conditions hold: $\ell_\gamma \subset G \cap \Omega$; the arcs $b_{21}$, $b_{22}$, $l_2$ do not have common interior points; the arcs $b_{21}$, $b_{22}$ join one of the endpoints (or both endpoints) of the arc $l_2$ with the endpoints of the arc $b_1$ and lie entirely in $G_2 \cap D$, with the exception of these endpoints which belong to the arc $b_1$.

Let $D_\gamma = (G_2 \cap D) \setminus \Omega$, $\ell_\gamma = \ell_\gamma \cup \ell_\gamma$. We denote by $\Gamma_\gamma = \Gamma(\ell_\gamma, \ell_\gamma, D_\gamma)$ the family of curves in $D_\gamma$, separating in $D_\gamma$ the arc $l_\gamma$ from $\partial D_\gamma \setminus \mathbb{C}$.

We define in $D_j$ ($j = 1, 2$) a Lebesgue measurable nonnegative function $\rho(z)$ an admissible metric for the family $\Gamma_j = \Gamma(\ell_j, \ell_j, D_j)$ if for any curve $\gamma \in \Gamma_j$ we have the relation

$$\int_{\gamma} |\rho(z)| dz > 1.$$ 

By the modulus of the family $\Gamma_j$ we mean the quantity

$$m(\Gamma_j) = \inf \{ \int_{\gamma} \rho(z) dx \} ,$$

where the infimum is taken over all metrics $\rho(z)$ that are admissible for $\Gamma_j$.

We exhaust from within the domain $D$ by the domains $D_n$, where

$$D_n \subset D_n, \quad \ell_n \cup \Omega \subset D_n \quad (n = 1, 2, \ldots).$$

Here we assume that the domain $D$ is bounded by a finite number of nondegenerate, nonintersecting, piecewise analytic curves. We note that if $D_n = (D_n \cap G_n) \setminus \ell_n$, $D_2n = (D_n \cap G_n) \setminus \Omega$, $\Gamma_n = \Gamma(\ell_n, \ell_n, D_n)$ ($j = 1, 2$), then from the continuity of the modulus under the extension

of the family of curves (see [1, p. 219]) there follows the equality
\[ \lim_{n \to \infty} m(\Gamma_{jn}) = m(\Gamma_j). \]

Without loss of generality, we shall assume in the sequel that \( D_j \) is a domain (\( j = 1, 2 \)).
In this case let \( H_{jn} = H(\zeta, D_{jn}) \) be a function that is harmonic in \( D_{jn} \) and equal to 0 on \( \zeta_j \),
equal to 1 on \( \partial D_j \setminus \{\zeta_j \cup \zeta_i\} \), while the derivative along the normal of the function \( H(z, D_{jn}) \)
is equal to 0 on \( b_j \). Then
\[ p(z, \Gamma_{jn}) = \left| \frac{\text{grad } H_{jn}}{C(H_{jn})} \right|, \]
where
\[ C(H_{jn}) = \iint_{D_{jn}} \left| \frac{\text{grad } H_{jn}}{C(H_{jn})} \right|^2 \, dx \, dy, \]
is the extremal metric [2] in the modulus problem for the family \( \Gamma_{jn} \). Moreover, \( m(\Gamma_{jn}) = C(H_{jn}) \).

Since \( 0 \leq H(z, D_{jn}) \leq 1 \) in \( D_{jn} \), it follows that the sequence \( \{H(z, D_{jn})\} \) converges uniformly inside \( D_j \) to the harmonic function \( H_j = H(\zeta, D_j) \) (see [3, p. 131]). Moreover, \( H_j = 0 \) on \( \zeta_j \), \( H_j = 1 \) on the set of all those points which belong to \( D_j \setminus \{\zeta_j \cup \zeta_i\} \) and are regular boundary points of the domain \( D_j \). Since the set \( \partial D_j \setminus \{\zeta_j \cup \zeta_i\} \) contains regular boundary points of the domain \( D_j \), it follows that \( H(z, D_j) \neq \text{const} \) in \( D_j \). Therefore, \( C(H_j) 
eq 0 \), \( m(\Gamma_j) = 1/C(H_j) \) and the function
\[ p(z, \Gamma_j) = \left| \frac{\text{grad } H_j}{C(H_j)} \right| \]
is extremal in the modulus problem for the family \( \Gamma_j \).

Remark 1.1. We mention that if \( m(\Gamma_j) < +\infty \), then the extremal metric \( p(\zeta, \Gamma_j) \) can be written effectively with the aid of the function \( H(z, D_j) \) also in the case when \( D_j \) is an open disconnected set.

Remark 1.2. The set \( D_j \) with a fixed pair of sides \( b_{j1}, b_{j2} \) can be considered as a generalization of a simply connected quadrangle [4], while the quantity \( m(\Gamma_j) \) is the modulus of the generalized quadrangle \( D_j \).

2. O-Univalent Functions and the Principle of R-Transform

1. Definition 2.1. Let \( D \) be an arbitrary domain of the plane \( \mathbb{C}_\zeta \) and let \( \zeta \in D \). A function \( w = f(z) \), regular in the domain \( D \), is said to be O-univalent if the following conditions hold: 1) \( f(\zeta) = 0 \); 2) there exists a neighborhood \( O(\zeta) \subset D \) of the point \( z = 0 \), in which \( w = f(z) \) is univalent and
\[ \{ w : w = f(z), z \in D \setminus O(\zeta) \} \cap \{ w : w = f(z), z \in O(\zeta) \} = \emptyset. \]

In the sequel we assume that \( w = f(z) \) is a 0-univalent function in the domain \( D \) and that in the definition of the family \( \Gamma_j (j = 1, 2) \) (see Sec. 1) the condition \( \zeta_j \subset O(\zeta) \) holds.

Let \( \{K\} = \{w : w = f(z), z \in K\} \), where \( K \subset D \); let \( \overline{F} \) be the closure of the set \( F \subset \mathbb{C}_\zeta \) in the topology of the extended plane \( \overline{\mathbb{C}}_\zeta \). We set
\[ B_j = \{\zeta_j\}, \quad L_j = \{\zeta_j\}, \quad D_j(f) = f(D_j) \setminus L_j. \]
By \( \Gamma_j(f) \) we denote the family of the curves in \( D_j(f) \cup B_j \), which separate in \( D_j(f) \) the curve \( L_j \) from \( \partial D_j(f) \setminus (B_j \cup L_j) \).

Let \( \{D_n\} \) be some fixed sequence of domains (see Sec. 1), exhausting from within the domain \( D \), \( D(n) \subset D \), \( n = 1, 2, \ldots \). Then the sequence \( \{f(D_n)\} \) will exhaust from within the domain \( f(D) \) and, extending, if necessary, the domain \( D_n \), one can assume that the domain \( f(D_n) \) is