THREE-DIMENSIONAL REGIONS OF STABILITY ABOUT 
THE TRIANGULAR EQUILIBRIUM POINTS

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(Received 14 July, 1982; Accepted 7 March, 1983)

Abstract. Within the context of the restricted problem of three bodies, an analytic upper bound on the 
three-dimensional regions of stability about the triangular equilibrium points is derived for general initial 
velocity limits and a wide class of bounding regions. This upper bound is illustrated and compared to 
numerical investigations for two bounding regions using the Earth–Moon mass ratio.

1. Introduction

Let two masses, \( m_1 \) and \( m_2 \), termed the primaries, influenced solely by their mutual 
Newtonian gravitational attraction, each move on a circular orbit about their center 
of mass. Consider a third mass, \( m_3 \), that is influenced by the primaries, but is suffi-
ciently small that its effect on the primaries is negligible. The restricted problem of 
three bodies is to describe the motion of \( m_3 \).

It was shown by Lagrange (1772) that there exist five equilibrium solutions to the 
restricted problem. Linearization about these points reveals that the three collinear 
points are unstable and the triangular points, denoted by \( L_4 \) and \( L_5 \), are neutrally 
stable for \( \mu = m_2/(m_1 + m_2) < \mu_0 \equiv \frac{1}{2}[1 - \frac{1}{2}(69)^{1/2}] = 0.038520896 \) (for example, 
see Szebehely, 1967). Leontovic (1962) proved that the triangular points are stable 
equilibrium solutions of the nonlinear system for \( \mu < \mu_0 \) except for a set of measure 
zero. Deprit and Deprit-Bartholome (1967) then showed that this set of measure 
zero contains only the three mass ratios corresponding to \( 2:1 \) and \( 3:1 \) commensur-
ability of the frequencies obtained by the linear analysis and one other value. The 
respective numerical values of these exceptional mass ratios are:

\[
\begin{align*}
\mu_2 &= \frac{1}{2}[1 - (1/45)(1833)^{1/2}] = 0.024293897, \\
\mu_3 &= \frac{1}{2}[1 - (2/45)(117)^{1/2}] = 0.013516016, \\
\mu_c &= 0.0191.
\end{align*}
\]

Subsequently, Markeev (1969) and Alfriend (1970, 1971) have shown that the tri-
angular equilibrium points are unstable for \( \mu = \mu_2, \mu = \mu_3 \), and stable for \( \mu = \mu_c \).

The articles of Leontovic (1962), Deprit and Deprit–Bartholome (1967), Markeev 
(1968), and Alfriend (1970, 1971) have all contributed to the understanding of the 
stability of the triangular equilibrium points without addressing the problem of 
determining the actual extent of the stability of these points. It is of additional interest, 
and the purpose of this article, to determine regions in the phase space about the
triangular equilibrium points in which solutions librate about the equilibrium points.

McKenzie and Szebehely (1979) employed numerical integration to investigate the planar configuration space about \( L_4 \) and \( L_5 \) for the Earth–Moon mass ratio \( (\mu = 0.0121409319) \). Their results establish regions about the triangular points which are remarkably similar in shape to certain zero velocity curves associated with the Jacobian integral. By numerically integrating the equations of motion of the restricted problem they have determined that a solution which originates, with zero velocity relative to the primaries, in these regions remains in the neighborhood of the triangular points for some finite time.

In the following it is shown that the Jacobian integral implies an analytic upper bound on the three dimensional regions of initial conditions which result in bounded solutions about the triangular equilibrium points for arbitrary initial velocity limits and a wide class of bounding regions. Conversely, this criterion defines regions of initial conditions which do not permit librational motion about the triangular equilibrium points.

This upper bound is shown to analytically predict the numerical results of McKenzie and Szebehely. The method is further used to derive an upper bound on the hitherto undetermined three dimensional regions of stability about the triangular equilibrium points. This upper bound is then refined by a numerical procedure similar to that of McKenzie and Szebehely.

2. The Dynamical System and the Jacobian Integral

The coordinate system used is the synodic, or rotating, coordinate system with origin at the center of mass of the primaries and x-axis coincident to the line of syzygies. The unit of distance is chosen to be the distance between the primaries and the unit of time is chosen to be \( P/2\pi \) where \( P \) is the orbital period of the primaries (see Figure 1).

With respect to this coordinate system, the equations of motion for \( m_3 \) are:

\[
\begin{array}{c}
\ddot{x} - 2\dot{y} = \frac{\partial \Omega}{\partial x} \\
\ddot{y} + 2\dot{x} = \frac{\partial \Omega}{\partial y} \\
\ddot{z} = \frac{\partial \Omega}{\partial z}
\end{array}
\] (1)

where

\[
\begin{align*}
\Omega(x, y, z) &= \frac{1}{2}(x^2 + y^2) + \frac{1 - \mu}{r_1} + \frac{\mu}{r_2} + \frac{1}{2}\mu(1 - \mu), \\
r_1^2 &= (x - \mu)^2 + y^2 + z^2, \\
r_2^2 &= (x - \mu + 1)^2 + y^2 + z^2,
\end{align*}
\]