Polarization Operators for $s = 1$

By

H. HØGAASEN

(Received November 9, 1963)

Spin projection operators, useful in the relativistic Hamiltonian description of vector particles are given.

A characteristic feature of relativistic equations is that the spin operator does not commute with the Hamiltonian. It is therefore of interest to find operators commuting with $H$ that removes the $2s+1$ fold degeneracy after the charge and momentum is quantized. For $s = \frac{1}{2}$ there is a quite extensive literature on the subject\textsuperscript{1,2}, and we wish to give some operators corresponding to the one given by STECH\textsuperscript{3} and KOPPE\textsuperscript{4} for electrons in the case of $s = 1$ particles.

To simplify the procedure we take the Duffin-Kemmer-Petiau equation in its reduced form\textsuperscript{5}

\[
H = (\tau_3 + i \tau_1) - \frac{1}{2m} \bar{p}^2 - \frac{i}{m} (s \cdot \bar{p})^2 + \tau_3 \bar{m} = \frac{1}{2m} [\tau_3 (E^2 + m^2) - i \tau_1 \bar{p}^2 Q],
\]

where

\[
E^2 = m^2 + \bar{p}^2, \quad Q = 2 \left( \frac{s \cdot \bar{p}}{\bar{p}} \right)^2 - 1,
\]

\(\tau_1, \tau_2, \tau_3\) is a set of Pauli matrices, \(s_i\) are the spin matrices for \(s = 1\) and \(\tau_3\) means the outer product \(\tau \otimes s\). From the property $H^2 = E^2$ it is then easy to show that a unitary $U$ (with respect to the metric $\tau_3$)\textsuperscript{6,7} can be found that leads to two "uncoupled" equations with well defined charge. With

\[
U = \frac{1}{2 \sqrt{E \bar{m}}} \left[ E + m + \tau_2 (E - m) Q \right] U^\dagger = \frac{1}{2 \sqrt{E \bar{m}}} \left[ E + m - \tau_2 (E - m) Q \right]
\]

\textsuperscript{3} STECH, B.: Z. Physik 144, 216 (1956).
we get
\[ H^T = U H U^\dagger = \tau_3 \left[ p^2 + m^2 \right]^{\frac{1}{2}} \quad \text{and} \quad P_\pm = \frac{1}{2} (1 \pm \tau_3) \]
where \( P_\pm \) is the usual charge projection operator.

In this new representation the spin operator \( s \) as well as \( \tau_3 s = s \) are constants of motion, as well as all linear combinations (also containing \( s \) operators) of these. \( s n \) and \( \tau_3 (s n) \) when \( n \) is an arbitrary unit vector can be quantized and is an operator inside the manifolds of positive and negative charge states separately.

In the original representation these operators have the form
\[ \Sigma_1 \cdot n = U^\dagger \tau_3 (s n) U \quad \text{and} \quad \Sigma_2 \cdot n = U^\dagger (s n) U \]
where \( \Sigma_2 \) is the operator corresponding to the Foldy-Wouthuysen mean spin, \( \Sigma_1 \) the operator defined by Stech and
\[ \frac{1}{4E m} \left\{ (E+m)^2 s n - (E-m)^2 Q(s n) Q \right\} = \Sigma_3 n \]
corresponds to Koppe's operator. We write the two first out explicitly:
\[ \Sigma_1 \cdot n = \frac{1}{4E m} \left[ \tau_3 \left\{ (E+m)^2 (s n) + (E-m)^2 Q(s n) Q \right\} - i \tau_1 p^2 \left( (s n) Q + Q(s n) \right) \right], \]
\[ \Sigma_2 \cdot n = \frac{1}{4E m} \left[ (E+m)^2 (s n) - (E-m)^2 Q(s n) Q + \tau_2 p^2 \left( (s n) Q - Q(s n) \right) \right] \]
The algebra of the operators \( \Sigma_1 \) and \( \Sigma_2 \) is
\[ \Sigma_i \Sigma_j \Sigma_k + \Sigma_k \Sigma_j \Sigma_i = \delta_{ij} \Sigma_k + \delta_{jk} \Sigma_i, \]
\[ (\Sigma_1 n)^2 = (\Sigma_2 n)^2 = \frac{1}{4m E} \left\{ (E+m)^2 (s n)^2 - (E-m)^2 Q(s n)^2 Q + p^2 \tau_2 [(s n)^2 Q - Q(s n)^2] \right\} \]
\[ = (s n)^2 - \frac{i}{2m E} \left[ (E-m)^2 Q - p^2 \tau_2 \right] \cdot (\hat{p})(s n \times \hat{p}) \]
where \( \hat{p} = p/|p| \).
\[ [[(\Sigma_1)_i, (\Sigma_1)_j]] = i \varepsilon_{ijk} \frac{H}{E} (\Sigma_1)_k, \]
\[ [[(\Sigma_2)_i, (\Sigma_2)_j]] = i \varepsilon_{ijk} (\Sigma_2)_k. \]
It is easily seen that \( (\Sigma n)^3 = \Sigma n \) and that the unit element in the space over which the spin operators operate can be written as the sum of three

---