Some problems of „Partitio numerorum": II. Proof that every large number is the sum of at most 21 biquadrates.

By

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1. Introduction.

1.1. This memoir is essentially a sequel to one which we published recently in the Göttinger Nachrichten[1]). It could not in any case be intelligible to a reader unacquainted with our earlier memoir; and we shall therefore quote formulae from the latter without further explanation.

In the memoir referred to we laid the foundations of our new method for the solution of Waring's Problem, carrying our analysis just so far as was necessary for the proof of Hilbert's Theorem, the fundamental existence theorem for the numbers $g(k)$ and $G(k)$. Here our object is to find the best possible inequality for the particular number $G(4)$. A good deal of our analysis, however, is valid for a general k, and will be useful to us when we proceed to the corresponding general problem. It will be found that the special interest of the case $k=4$ is quite sufficient to justify its consideration in a separate memoir.

1.2. It is known that

$$19 \leqq g(4) \leqq 37, \qquad 16 \leqq G(4) \leqq 37,$$

these inequalities, from left to right, being due to Waring, Wieferich, Kempner, and Wieferich respectively. For detailed references we may refer to the dissertations of Kempner[2]) and of Baer[3]). We need men-

[1]) G. H. Hardy and J. E. Littlewood, Some problems of 'Partitio numerorum'; I: A new solution of Waring's Problem, Göttinger Nachrichten 1920, S. 33–54. We shall refer to this memoir as W. P.

[2]) A. J. Kempner, Über das Waringsche Problem und einige Verallgemeinerungen, Inaugural-Dissertation, Göttingen 1912.

[3]) W. S. Baer, Beiträge zum Waringschen Problem, Inaugural-Dissertation, Göttingen 1913.

tion only that the deepest result, viz. $g(4) \leqq 37$, was obtained in 1909 by Wieferich, whose analysis is a refinement upon that by which Landau, in 1907, had proved that $g(4) \leqq 38$. Here we shall prove nothing concerning $g(4)$; but we shall improve the upper bound for $G(4)$ very notably, by proving

Theorem A: $G(4) \leqq 21$.

2. A sharpening of our earlier analysis.

2.1. In § 9.2. of W. P. we proved that, assuming always

$$s \geqq 2K + 1 = 2^k + 1,$$

$$(2.11) \qquad \begin{aligned} r_{k,s}(n) &= C n^{s\alpha-1} S + O(n^{s\alpha\varkappa+\varepsilon}) + O(n^{s\alpha+\alpha\varkappa-\alpha-1+\varepsilon}) \\ &= \varrho_{k,s}(n) + O(n^{s\alpha\varkappa+\varepsilon}) + O(n^{s\alpha+\alpha\varkappa-\alpha-1+\varepsilon}), \end{aligned}$$

where

$$S = \sum \left(\frac{S_{p,q}}{q}\right)^s e_q(-np).$$

It will be necessary now to replace the term $O(n^{s\alpha\varkappa+\varepsilon})$ by a term of lower order[4]).

2.2. It will be found, on an examination of the analysis of W. P., that the critical error term $O(n^{s\alpha\varkappa+\varepsilon})$ arises in two places only. All other errors are of lower order than that of the dominant factor $n^{s\alpha-1}$, either independently of the value of s, or at any rate when $s \geqq 2K+1$. The two critical errors arise as follows.

In the first place we have

$$S_2 = \sum \frac{1}{2\pi i} \int\limits_{\mathfrak{m}} \frac{(f(x))^s}{x^{n+1}}\, dx = O(n^{s\alpha\varkappa+\varepsilon}),$$

where $\mathfrak{m}$ is a typical minor arc of the Farey dissection.

Secondly, when we consider the corresponding sum connected with the major arcs, we are confronted by a sum $\sum \sigma_{p,q}$, where

$$\sigma_{p,q} = \int\limits_{\mathfrak{M}} \frac{|\Phi|^s}{|x|^{n+1}}\, |dx|$$

and $\mathfrak{M}$ is a typical major arc of the dissection, and we write

$$\sum \sigma_{p,q} = O(n^{s\alpha\varkappa+\varepsilon}).$$

It will be observed that these two errors arise in exactly the same way. The upper bounds are obtained by substituting in the integrals the crude approximations, $f = O(n^{\alpha\varkappa+\varepsilon})$ on a minor arc and $\Phi = O(n^{\alpha\varkappa+\varepsilon})$

[4]) The formula (2.11) would lead only to $G(4) \leqq 33$, in itself a new result.