Approximation of a Nondifferentiable Nonlinear Problem Related to MHD Equilibria

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Summary. We present a simple method, based on a variant of the implicit function theorem, which leads to the existence of (a part of) a nontrivial solution branch of the nonlinear eigenvalue problem \(-\Delta u = \lambda u^+\) in \(\Omega\), \(u = -1\) on \(\partial\Omega\), where \(\Omega\) is a two-dimensional domain with boundary \(\partial\Omega\). The advantage of this method is that we can apply it for analysing the approximation of the above problem by a finite element method; the error analysis of the discrete problem appears immediately. We give also an iteration scheme which allows to solve the approximate problem.


1. Introduction

Let \(\Omega\) be a bounded and connected domain of \(\mathbb{R}^2\) with a Lipschitzian boundary \(\partial\Omega\). We are interested in a function \(u \in H^1(\Omega)\) and a real number \(\lambda\) such that

\begin{align*}
-\Delta u &= \lambda u^+ \quad \text{in } \Omega \\
u &= -1 \quad \text{on } \partial\Omega
\end{align*}

(1.1)

where \(\Delta\) is the Laplacian operator and \(u^+\) is the function defined by \(u^+(x) = u(x)\) if \(u(x) \geq 0\) and \(u^+(x) = 0\) if \(u(x) < 0\). Clearly, the constant function \(u(x) = -1\) satisfies (1.1) for all \(\lambda \in \mathbb{R}\) and we are looking for other solutions of (1.1).

Several authors have treated Problem (1.1) which is related to ideal MHD equilibria in torus. For example Temam [12], Berestycki and Brézis [1], Puel [10] prove the existence of nontrivial solutions by variational methods. In [8], Kikuchi constructs a path of solutions of (1.1) by an iterative method which is based on the principle of contracting mappings.

In this paper, we adopt the point of view of Kikuchi [8] in connection with a variant of the implicit function theorem stated by Girault and Raviart in [5]. In fact, by adding a convenient normalization equation to the transformation of Problem (1.1) performed by Kikuchi, we can apply a result of [5] in order
to prove the existence of a path of solutions of Problem (1.1). We show that the advantage of this technique is that we can immediately perform the mathematical analysis of a finite element method for approximating Problem (1.1) and we obtain a very simple error analysis. Moreover, this point of view leads to a natural iterative method for solving the finite dimensional approximation of Problem (1.1). Let us mention here that our iterative scheme is very close to the one proposed by Georg in [4] or Descloux and Rappaz in [3] and is simpler than Kikuchi's method. For the mathematical and numerical study of this kind of problems, we mention furthermore the work of Sermange [11].

An outline of the paper is as follows. In Sect. 2, we state a simple generalization of the implicit function theorem in a nondifferentiable case; essentially we follow the proof of [5]. Section 3 is devoted to Problem (1.1) and Sect. 4 to a finite dimensional approximation of (1.1) in which we apply the abstract results of Sect. 2. In Sect. 5, we prove the convergence of an iterative method for solving the approximate problem.

Throughout the paper, we shall constantly use the following notations. For two Banach spaces X and Y we shall denote by \( \mathcal{L}(X; Y) \) the Banach space of all the continuous linear operators \( T \) from \( X \) into \( Y \). We shall use the classical Sobolev spaces \( W^{m,p}(\Omega) \) and \( H^m(\Omega) = W^{m,2}(\Omega) \) with the norms \( \| \cdot \|_{m,p,\Omega} = \| \cdot \|_{W^{m,p}(\Omega)} \) and \( \| \cdot \|_{m,\Omega} = \| \cdot \|_{m,2,\Omega} \); we denote by \( H^1_0(\Omega) = \{ f \in H^1(\Omega) : f = 0 \text{ on } \partial \Omega \} \).

2. An Implicit Function Theorem

In order to obtain a variant of the implicit function theorem stated in [5], Lemma 2, for a class of "almost" differentiable functions, we recall the inverse function theorem in the form given in [5], Lemma 1.

Let \( X \) and \( Y \) be two Banach spaces and \( f \) be a continuous mapping from a neighborhood of a point \( x_0 \in X \) into \( Y \); we set \( y_0 = f(x_0) \). In the following we denote by \( \| \cdot \| \) the various norms in the spaces \( X, Y, \mathcal{L}(X; Y), \mathcal{L}(Y; X) \) and by

\[
B(x_0, \delta) = \{ x \in X : \| x - x_0 \| \leq \delta \}, \quad B(y_0, \varepsilon) = \{ y \in Y : \| y - y_0 \| \leq \varepsilon \}
\]

for \( \delta \) and \( \varepsilon \in \mathbb{R} \).

**Lemma 1.** Assume there exist positive constants \( \delta, M \) and an isomorphism \( A \in \mathcal{L}(X; Y) \) such that

\[
\| A^{-1} \| \leq M \tag{2.1}
\]

and for all \( x, x^* \in B(x_0, \delta) \)

\[
\| f(x) - f(x^*) - A(x - x^*) \| \leq \frac{1}{2M} \| x - x^* \|. \tag{2.2}
\]

Then there exists a unique continuous function \( g \) defined in \( B\left(y_0, \frac{\delta}{2M}\right) \) with range in \( B(x_0, \delta) \) such that

\[
f(g(y)) = y, \quad \forall y \in B\left(y_0, \frac{\delta}{2M}\right). \tag{2.3}
\]