Galerkin-wavelet methods for two-point boundary value problems

Jin-Chao Xu¹ and Wei-Chang Shann²

¹ Department of Mathematics, The Pennsylvania State University, University Park, PA 16802, USA
² National Central University, Chung-Li, Taiwan, R.O.C.

Received May 1, 1991

Summary. Anti-derivatives of wavelets are used for the numerical solution of differential equations. Optimal error estimates are obtained in the applications to two-point boundary value problems of second order. The orthogonal property of the wavelets is used to construct efficient iterative methods for the solution of the resultant linear algebraic systems. Numerical examples are given.

Mathematics Subject Classification (1991): 65N30, 65N13, 65F10

1. Introduction

In this paper, we shall study wavelet functions applying to the numerical solution of differential equations. Wavelets in our considerations are those discovered by Daubechies in 1988 [7], which have compact supports and form an orthonormal basis of \( L^2(\mathbb{R}) \).

The idea of wavelets may be traced back to Calderon [2] and Coifman and Weiss [4]. Modern ideas grow out of applications in signal processing, cf. Liénard [19], Rodet [24] and Goupillaud, Grossmann and Morlet [12]. Earlier mathematical analysis of wavelets can be found in Stromberg [26], Grossmann and Morlet [14] and Meyer [22]. In recent years, wavelets have been studied by Meyer and his group using the multiresolution framework, cf. Mallat [20] and Meyer [23]. The multiresolution approach has lead to the discovery of a family of wavelets with compact supports by Daubechies in 1988 [7]. For the application of wavelets to practical problems, we refer to [5, 6, 10, 13, 17, 18, 21].

Applying wavelets to discretize differential equations appears to be a very attractive idea. In finite element type methods, piecewise polynomial trial functions may

* This work was supported by National Science Foundation

Correspondence to: Jin-Chao Xu
be replaced by wavelets. Such an idea was explored by Glowinski, Lawton, Rava-
chol and Tenenbaum [10] (see also the unpublished technical reports cited therein).
Many interesting numerical examples in their paper suggest that wavelets have great
potential in the application to the numerical solution of differential equations.

At this stage of research, a thorough study of one dimensional problems is
still necessary and theoretically important. The simplest case would be the two-point
boundary value problem for the second order elliptic equation, but how the wavelets
can be appropriately used in this case is still not clear. If the wavelets are used directly
as trial functions, as is done in [10], several difficulties would be encountered. First,
due to the lack of regularity, “lower order” wavelets can not be employed. Secondly,
Dirichlet boundary value problems can not be applied directly because the boundary
conditions are hard to impose on subspaces. Thirdly, the orthogonality, one of the
main features of wavelets, does not play any significant roles.

In this paper, we shall take a different approach. Instead of using wavelets directly,
we take their anti-derivatives as trial functions. In this way, singularity in the wavelets
is smoothed, the boundary condition can be treated easily and the orthogonality is
used to construct efficient algorithms to solve the underlying algebraic system.

Our approach was motivated by the observation that the differentiation of the so-
called hierarchical basis functions (cf. [27, 28]) in one dimensional piecewise linear
finite elements are exactly the wavelets of order 1, namely the Haar basis [15]. Taking
the anti-derivatives of higher order wavelets then leads to higher order trial functions.

This work is our first attempt to use wavelets for numerical solution of differential
equations. For one-dimensional problems studied in this paper, the wavelets approach
appears to be attractive and our theory is rather complete. Applications to higher
dimensional problems (on the domains other than rectangles) remain to be explored
and the prospect of the method is yet to be seen.

We shall use the standard notation $L^2(\mathbb{R})$ to denote the space of square integrable
functions. Two functions $u, v \in L^2(\mathbb{R})$ are orthogonal in $L^2(\mathbb{R})$ if $(u, v) = 0$.
Given $\Omega = (a, b)$ ($-\infty < a < b < \infty$), $H^s(\Omega)$ denotes the standard Sobolev space with the
norm $\| \cdot \|_{s, \Omega}$ and semi-norm $| \cdot |_{s, \Omega}$ given by

$$
\| v \|_{s, \Omega}^2 = \sum_{i=0}^{s} \int_{a}^{b} |v^{(i)}(x)|^2 \, dx \quad \text{and} \quad |v|_{s, \Omega}^2 = \int_{a}^{b} |v^{(s)}(x)|^2 \, dx.
$$

For $-\infty < a < b < \infty$, we define

$$
H^1_0(\Omega) = \{ v \in H^1(\Omega) \mid v(a) = v(b) = 0 \} \quad \text{and} \quad H^1_+(\Omega) = \{ v \in H^1(\Omega) \mid v(a) = 0 \}.
$$

It is well-known that the semi-norm $| \cdot |_{1, \Omega}$ is a norm (equivalent to $\| \cdot \|_{1, \Omega}$) in these
two spaces. When $\Omega = \mathbb{R}$, we denote $\| v \|_s = \| v \|_{s, \Omega}$ and $|v|_s = |v|_{s, \Omega}$.

Throughout this paper, we shall use the letter $C$ to denote a generic positive
constant which may stand for different values at its different appearances. When it is
not important to keep track of these constants, we shall conceal the letter $C$ into the
notation $\preceq$, $\simeq$, or $\geq$. Here

(1.1) \hspace{1cm} x \preceq y \hspace{1cm} \text{means} \hspace{1cm} x \leq Cy,

$x \geq y$ means $-x \preceq -y$, and $x \simeq y$ means $x \preceq y$ and $x \geq y$.

The rest of the paper is organized as follows. Section 2 gives a brief introduction
to wavelet functions. In Sect. 3, a simple technique by means of anti-derivatives...