Note on the numerical integration of periodic functions and of partially periodic functions

By

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Recently there have appeared a number of papers dealing with the error estimation of certain approximate integration formulas for periodic functions of several variables (see [1]—[9]). The principal techniques employed in most of the papers cited are that of trigonometric approximations as originally proposed in [4], [5] and [1], respectively. Here we shall show that a sharpest estimation for an ordinary approximation formula may even be more directly attained by means of the Euler-Maclaurin summation formula. Moreover, this note is also concerned with the problem of reducing the numerical integration of partially periodic functions of several variables to that of functions with less variables.

In what follows we always denote for brevity \( X = (x_1, \ldots, x_m) \), \( Y = (y_1, \ldots, y_n) \), \( dX = dx_1 \ldots dx_m \), \( dY = dy_1 \ldots dy_n \), \( f(X) = f(x_1, \ldots, x_m) \), \( f(X, Y) = f(x_1, \ldots, x_m, y_1, \ldots, y_n) \). Moreover, \( N \) and \( R \) always stand for positive integers, and \( U \) and \( V \) denote the \( m \)-cell \( U(0 \leq x_1 \leq 1, \ldots, 0 \leq x_m \leq 1) \) and the \( n \)-cell \( V(0 \leq y_1 \leq 1, \ldots, 0 \leq y_n \leq 1) \), respectively.

We shall always assume \( f(X) \) and \( f(X, Y) \) to be periodic continuous functions having period 1 with respect to each of the variables \( x_1, \ldots, x_m \). In general \( f(X, Y) \) may be not periodic in \( y_i (1 \leq i \leq n) \) and is therefore called a "partially periodic function".

1. First let us consider the numerical computation of the integral

\[
I(f) = \int_U f(X) \, dX.
\]  

(1)

Here all the \( \varphi \)-th order partial derivatives \( \frac{\partial^\varphi f}{\partial x_i^\varphi} \) are assumed to exist and be continuous throughout \( U \) with

\[
\left| \frac{\partial^\varphi f}{\partial x_i^\varphi} \right| \leq M, \quad (i = 1, \ldots, m)
\]  

(2)

\( \varphi \) being a positive even integer and \( M \) a positive constant. It is now easy to prove the following

**Theorem 1.** Let \( N = R^m \) and express

\[
I(f) = \frac{1}{N} \sum_{r_1, \ldots, r_m} f \left( \frac{r_1}{R}, \ldots, \frac{r_m}{R} \right) + e_N.
\]  

(3)
Then the error term \( q_N \) has the estimate

\[
|q_N| \leq 2m M \left( \frac{\zeta(p)}{(2\pi)^p} \right) \left( \frac{1}{N} \right)^{\nu/m},
\]

(4)

where \( \zeta(p) \) is the \( \zeta \)-function of Riemann.

The estimation given by (4) is actually very sharp. That the order \( O(N^{-\nu/m}) \) is the best possible has already been noted by Solodov [9] and Min [6]. Explicit estimation has also been found by Min [6], using the method of trigonometric approximation. But Min’s result is inferior to (4), and it becomes meaningless for \( p \leq m \).

Here we sketch a proof of (4). By successive applications of the Euler summation formula (with \( R = R \quad R \ldots R \) as nodes) to the repeated integral \( I(f) \) and by means of the first mean-value theorem in the integral calculus we may finally find that the remainder \( q_N \), as defined by (3), can be expressed in the form

\[
q_N = - \left( \frac{1}{R} \right)^p \sum_{i=1}^{m} \left( \frac{\partial}{\partial x_i} \right)^p f(x_1^{(i)}, \ldots, x_m^{(i)}),
\]

(5)

where \( x_1^{(i)}, \ldots, x_m^{(i)} \) \((i = 1, \ldots, m)\) are certain points interior to \( U \), and \( B_p \) is known as the Bernoulli number with even index \( p \). Actually the expression (5) can also be verified by using induction on \( m \), the number of variables contained.

Since the evaluation involved is quite elementary, we may omit its details here.)

Now recalling the well-known relation between the Bernoulli number \( B_p \) and the function \( \zeta(p) \), we see that (5) is equivalent to

\[
q_N = (-1)^{\frac{1}{2}} \frac{2\zeta(p)}{(2\pi)^p} \left( \frac{1}{N} \right)^{\nu/m} \sum_{i=1}^{m} \left( \frac{\partial}{\partial x_i} \right)^p f(x_1^{(i)}, \ldots, x_m^{(i)}),
\]

Consequently we get the inequality (4) by making use of (2).

Remark. Since the Riemann sum on the right-hand side of (3) is performed by the method of uniform net, it may be seen from (4) that the Monte Carlo method, when applied to computing \( I(f) \), will surely yield very satisfactory results.

2. We now turn our attention to the numerical evaluation of the \((m + n)\)-fold integral

\[
J(f) = \int \int f(X, Y) dX dY.
\]

(6)

The function \( f(X, Y) \) is assumed to satisfy the differentiability condition of the same type as that imposed upon \( f(X) \); i.e. we have (2) with \( f(X) \) being replaced by \( f(X, Y) \), and \( U \) being replaced by \( U \times V \).

A result to be stated here is that the numerical evaluation of \( J(f) \) can always be reduced to that of a certain \((n + 1)\)-fold integral or of certain \( n \)-fold integrals.

Theorem 2. Let \( \gamma_1, \ldots, \gamma_m \) be a set of integers such that \( \gamma_1 > \gamma_2 > \cdots > \gamma_m \geq 0 \), and let \( F(t, Y) \equiv f(R^{\gamma_1}t, \ldots, R^{\gamma_m}t, y_1, \ldots, y_n) \). Then for \( R \) large we have

\[
J(f) = \int_{Y}^{1} \int_{0}^{1} F(t, Y) dt dY + O\left[ \left( \frac{1}{R} \right)^{p} \right].
\]

(7)