CONCERNING ALGEBRAIC INDEPENDENCE
OF SOME TRANSCENDENTAL NUMBERS

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Given the three numbers $a_1^\beta$, $a_2^\beta$, and $\ln a_2/\ln a_1$, where $a_1$ and $a_2$ are algebraic numbers whose logarithms are linearly independent in a rational field and $\beta$ is a quadratic irrationality, it is shown that they are not all expressible algebraically in terms of one of them.

In 1934, A. O. Gel'fond [1] established the transcendence of numbers of the form $a^\beta$, where $a \neq 0$, 1 and is algebraic and $\beta$ is an algebraic irrationality.

Let $\alpha_1, \ldots, \alpha_s$ be transcendental numbers and let $Q(x_1, \ldots, x_s)$ be a polynomial which is irreducible in a rational field and which has rational integer coefficients, the highest common divisor of which is unity. We shall say that the numbers $\alpha_1, \ldots, \alpha_s$ are algebraically independent in a rational field if there is no polynomial $Q$ for which $Q(\alpha_1, \ldots, \alpha_s) = 0$. We shall say also that the numbers $\alpha_1, \ldots, \alpha_s$ are expressible algebraically in terms of one of them if there exist $s$ equations

$$Q_k(x, \alpha_i) = 0, \quad k = 1, \ldots, s,$$

where all the $Q_k(x, \alpha_i) \neq 0$.

In 1948-1949, Gel'fond [2, 3] developed a new analytic method which he applied successfully to various questions relating to the transcendence of numbers. In this paper Gel'fond's method is used to prove a theorem concerning the algebraic independence of certain classes of numbers.

THEOREM. Let $\alpha_1$ and $\alpha_2$ be algebraic numbers whose logarithms are linearly independent in a rational field; let $\beta$ be a quadratic irrationality, and let $V = \ln \alpha_s/\ln \alpha_2$. Then, the three numbers $V, \alpha_1^\beta, a_2^\beta$ are not expressible algebraically in terms of one of them.

The transcendence of each of these numbers is assumed to be already known [1].

LEMMA 1 [4]. If $a_{kn}$ ($1 \leq k \leq r, 1 \leq n \leq s, r \geq 2s$) are rational integers, $|a_{kn}| \leq a$, then it is possible to find rational integers $x_1, \ldots, x_r$, $\sum_{k=1}^{r} x_k > 0$, such that

$$\sum_{k=1}^{r} a_{kn}x_k = 0 \quad (n = 1, \ldots, s),$$

wherein $|x_k| \leq 3ar$.

LEMMA 2 [3]. Let $\alpha \neq 0, a_0 > 1, \sigma(x) > x, \theta(x) > 0$ be given, where, for $x > x_0 > 0$, $\sigma(x)$ and $\theta(x)$ are monotonic and grow unboundedly with $x$, and $a_0\sigma(x) \approx a(x + 1)$. Then, if for an arbitrary integer $N > N_0 > 0$, there exists a polynomial $P(x) \neq 0$ with rational integer coefficients of height $H$ and of degree $n$, satisfying the condition

$$|P(x)| \ll e^{-\sigma(n)(N)} \quad \text{max}[n,\ln H] \leq \frac{1}{\sigma(N)},$$

the number $\alpha$ must be algebraic.

LEMMA 3 [3]. Let $p, q, p < q\gamma$, $r, r_1$ be positive integers, $\varepsilon > 0$, and let $\gamma$ be fixed; assume also that the numbers $\alpha_1, \ldots, \alpha_q$, as well as the numbers $\beta_1, \ldots, \beta_r$, are mutually distinct and ordered according to decreasing moduli, i.e., $|\alpha_k| \leq |\alpha_{k+1}|$ and $|\beta_k| \leq |\beta_{k+1}|$. We put $|\alpha_q| = \alpha$, $|\beta_r| = \beta$ and assume the existence of constants $\gamma_0 > 0, \gamma_1 > 0, \gamma_0 + \gamma_1 < 1$, such that $\alpha < (pq)^{\gamma_1}$, $\beta < (pq)^{\gamma_0}$. We assume also the

existence of a constant $\gamma_2$ such that the inequality

$$\prod_{k=1}^{q} |x_k - x_i| > e^{-\gamma_2 \ln q}, \quad |x_1 - x_k| > e^{-\gamma_2 \ln q},$$

\[1 \leq i \leq q, \quad 1 \leq k \leq q\]

is satisfied. Then, if the function $f(z)$ has the form

$$f(z) = \sum_{s=0}^{\infty} \frac{A_k z^s}{s!},$$

where the numbers $A_k$ are in totality distinct from zero, at least one of the numbers

$$f^{(s)}(\beta), \quad 0 \leq s \leq r_1 - 1, \quad 1 \leq k \leq r,$$

$$r_1 r \geq [\lambda pq], \quad \lambda = \frac{1 + \gamma_1 + 2\gamma_2 + \epsilon}{1 - \gamma_1 - \gamma_2},$$

is distinct from zero for sufficiently large $pq$.

Proof of the Theorem. Assume that the theorem is false; then there exists a transcendental number $\omega$ such that

$$P_1(\omega, \eta) = 0, \quad P_2(\omega, \alpha) = 0, \quad P_\mu(\omega, \alpha) = 0,$$

$$P_0(x, y) = P_1(x, y) P_2(x, y) P_\mu(x, y) \equiv 0,$$

where $P_1(x, y)$, $P_2(x, y)$, $P_\mu(x, y)$ are irreducible polynomials in two independent variables with rational integer coefficients. Let $R_0 = R(\omega)$ be the extension of the field of rational numbers obtained by adjoining to it the transcendental number $\omega$. Let the polynomial $P(\omega, y)$ be of degree $\mu$ in $y$; denote the roots of the equation $P_0(\omega, y) = 0$ by $\alpha_1, \ldots, \alpha_\mu$. We indicate by $R_1 = R_0(\omega_1)$ the smallest algebraic extension of the field $R_0$ which contains the numbers $\alpha_1, \ldots, \alpha_\mu$; by $\nu$, the degree of this field; by $\omega_1$, its primitive element; and by $\omega_2, \ldots, \omega_\nu$ the conjugates of $\omega_1$ in the field $R_0$. If

$$P_0(x, y) = \sum_{k=0}^{\nu} \sum_{l=0}^{\mu} a_{kl} x^k y^l,$$

then let

$$T(\omega) = \sum_{k=0}^{\nu} \sum_{l=0}^{\mu} a_{kl} \omega^k, \quad S_i = T\alpha_i \quad (i = 1, \ldots, \nu).$$

It is evident that the $S_i$ are the integers of the field $R_1$; therefore,

$$S_i = \sum_{k=0}^{\nu} a_{ik} \omega_k,$$

where the $P_{i,k}(\omega)$ are certain polynomials of $\omega$ with rational integer coefficients. Let the largest of the degrees of the polynomials $P_{i,k}(\omega)$ be $m_2$. Since $S_i$ are the integers of the field $R_1$, then

$$|S_i|^{\mu} = \sum_{k=0}^{\nu} B_{ik} \omega_k^k \quad (i = 1, \ldots, \nu),$$

where $B_{ik}$ are the polynomials in $\omega$ with rational integer coefficients. Considering the numbers conjugate to $S_i$ with respect to the field $R_0$, we obtain

$$S_i^{(r)} = \sum_{k=0}^{\nu} B_{ik} \omega_k^k \quad (r = 1, \ldots, \nu),$$

where $S_i^{(r)}$. We consider now the relation (9) as a system of linear equations for determining the $B_{ikl}$. Since the determinant of the system of equations (9) is a Vandermonde determinant, then

$$\prod_{i=1}^{\nu} |B_{i,k,l}| \leq e^{\gamma_0}, \quad \gamma_0 > 0,$$