CONCERNING ALGEBRAIC INDEPENDENCE
OF SOME TRANSCENDENTAL NUMBERS

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Given the three numbers \( a_1^\beta, a_2^\beta, \) and \( \ln a_2/\ln a_1 \), where \( a_1 \) and \( a_2 \) are algebraic numbers whose logarithms are linearly independent in a rational field and \( \beta \) is a quadratic irrationality, it is shown that they are not all expressible algebraically in terms of one of them.

In 1934, A. O. Gel'fond [1] established the transcendence of numbers of the form \( a^\beta \), where \( a \neq 0, 1 \) and is algebraic and \( \beta \) is an algebraic irrationality.

Let \( \alpha_1, \ldots, \alpha_s \) be transcendental numbers and let \( Q(x_1, \ldots, x_s) \) be a polynomial which is irreducible in a rational field and which has rational integer coefficients, the highest common divisor of which is unity. We shall say that the numbers \( \alpha_1, \ldots, \alpha_s \) are algebraically independent in a rational field if there is no polynomial \( Q \) for which \( Q(\alpha_1, \ldots, \alpha_s) = 0 \). We shall say also that the numbers \( \alpha_1, \ldots, \alpha_s \) are expressible algebraically in terms of one of them if there exist \( s \) equations

\[
Q_k(x_k, \alpha_i) = 0, \quad k = 1, \ldots, s, \tag{1}
\]

where all the \( Q_k(x, \alpha_i) \neq 0 \).

In 1948-1949, Gel'fond [2, 3] developed a new analytic method which he applied successfully to various questions relating to the transcendence of numbers. In this paper Gel'fond's method is used to prove a theorem concerning the algebraic independence of certain classes of numbers.

**THEOREM.** Let \( \alpha_1, \alpha_2 \) be algebraic numbers whose logarithms are linearly independent in a rational field; let \( \beta \) be a quadratic irrationality, and let \( \eta = \ln a_1/\ln a_2 \). Then, the three numbers \( \eta, \alpha_1^\beta, \alpha_2^\beta \) are not expressible algebraically in terms of one of them.

The transcendence of each of these numbers is assumed to be already known [1].

**LEMMA 1 [4].** If \( a_{kn} \) \( (1 \leq k \leq r, 1 \leq n \leq s, r \geq 2s) \) are rational integers, \( |a_{kn}| \leq a \), then it is possible to find rational integers \( x_1, \ldots, x_r, \sum_{k=1}^{r} x_k^2 > 0 \), such that

\[
\sum_{k=1}^{r} a_{kn} x_k = 0 \quad (n = 1, \ldots, s), \tag{2}
\]

wherein \( |x_k| \leq 3a_r \).

**LEMMA 2 [3].** Let \( \sigma > 0, \sigma_0 > 1, \sigma \sigma_0 > \epsilon > 0 \) be given, where, for \( x > x_0 > 0, \sigma \sigma_0 \) and \( \sigma_0 \) are monotonic and grow unboundedly with \( x \), and \( \sigma \sigma_0 (x) \geq \sigma (x + 1) \). Then, if for an arbitrary integer \( N > N_0 > 0 \), there exists a polynomial \( P(x) \neq 0 \) with rational integer coefficients of height \( H \) and of degree \( n \), satisfying the condition

\[
|P(x)| \leq e^{\epsilon N H(N)}, \quad \max \{|n, \ln H| \leq \epsilon \sigma (N), \tag{3}
\]

the number \( \sigma \) must be algebraic.

**LEMMA 3 [3].** Let \( p, q, p < q \) be positive integers, \( \epsilon > 0 \), and let \( \gamma \) be fixed; assume also that the numbers \( \alpha_1, \ldots, \alpha_q \), as well as the numbers \( \beta_1, \ldots, \beta_r \), are mutually distinct and ordered according to decreasing moduli, i.e., \( |\alpha_k| \leq |\alpha_{k+1}| \) and \( |\beta_k| \leq |\beta_{k+1}| \). We put \( |\alpha_q| = \alpha, |\beta_r| = \beta \) and assume the existence of constants \( \gamma_0 > 0, \gamma_1 > 0, \gamma_0 + \gamma_1 < 1 \), such that \( \alpha < (pq)^{\gamma_1}, \beta < (pq)^{\gamma_0} \). We assume also the

existence of a constant $\gamma_2$ such that the inequality

$$\prod_{k=1}^{q} \left| x_k - a_i \right| > e^{-\gamma_2 \log q}, \quad \left| a_1 - a_k \right| > e^{-\gamma_2 \log q},$$

$$1 \leq i \leq q, \quad 1 \leq k \leq q$$
is satisfied. Then, if the function $f(z)$ has the form

$$f(z) = \sum_{q=0}^{P-1} \sum_{j=0}^{p-1} A_{k+j} e^{\pi i j},$$

where the numbers $A_{k+j}$ are in totality distinct from zero, at least one of the numbers

$$f^{(j)}(z), \quad 0 \leq s \leq r - 1, \quad 1 \leq k \leq r,$$

$$r_1 r > [\lambda pq], \quad \lambda = \frac{1 + \gamma_1 + 2\gamma_2 + \varepsilon}{1 - \gamma_1 - \gamma_2},$$
is distinct from zero for sufficiently large $pq$.

**Proof of the Theorem.** Assume that the theorem is false; then there exists a transcendental number $\omega$ such that

$$P_1(\omega, \eta) = 0, \quad P_2(\omega, a_1^\eta) = 0, \quad P_3(\omega, a_1^\eta) = 0,$$

$$P_0(z, y) = P(z, y), \quad P(z, y), \quad P(z, y) \neq 0,$$

where $P_1(x, y), P_2(x, y), P_3(x, y)$ are irreducible polynomials in two independent variables with rational integer coefficients. Let $R_0 = R(\omega)$ be the extension of the field of rational numbers obtained by adjoining to it the transcendental number $\omega$. Let the polynomial $P(z, y)$ be of degree $\mu$ in $y$; denote the roots of the equation $P_0(\omega, y) = 0$ by $\alpha_1, \ldots, \alpha_{\mu}$. We indicate by $R_1 = R_0(\omega_1)$ the smallest algebraic extension of the field $R_0$ which contains the numbers $\alpha_1, \ldots, \alpha_{\mu}$; by $\nu$, the degree of this field; by $\omega_1$, its primitive element; and by $\omega_2, \ldots, \omega_\nu$ the conjugates of $\omega_1$ in the field $R_0$. If

$$P_0(z, y) = \sum_{k=0}^{m_1} \sum_{l=0}^{m_2} a_k z^k y^l,$$

then let

$$T(\omega) = \sum_{k=0}^{m_1} a_{km} \omega^k, \quad S_i = T\alpha_i \quad (i = 1, \ldots, \mu).$$

It is evident that the $S_i$ are the integers of the field $R_1$; therefore,

$$S_i = \sum_{l=0}^{\nu-1} P_{ik}(\omega) \omega_1^k,$$

where the $P_{ik}(\omega)$ are certain polynomials of $\omega$ with rational integer coefficients. Let the largest of the degrees of the polynomials $P_{ik}(\omega)$ be $m_2$. Since $S_i$ are the integers of the field $R_1$, then

$$(S_i)^t = \sum_{l=0}^{\nu-1} B_{ikl} \omega_1^k \quad (i = 1, \ldots, \mu),$$

where $B_{ikl}$ are the polynomials in $\omega$ with rational integer coefficients. Considering the numbers conjugate to $S_i$ with respect to the field $R_0$, we obtain

$$(S_i)^t = \sum_{l=0}^{\nu-1} B_{ikl} \omega_1^k \quad (r = 1, \ldots, \nu),$$

where $S_i = S_i^t$. We consider now the relation (9) as a system of linear equations for determining the $B_{ikl}$. Since the determinant of the system of equations (9) is a Vandermonde determinant, then

$$| B_{i,k,l} | \leq e^{\gamma_2}, \quad \gamma_2 > 0,$$