ASYMPTOTIC SOLUTION OF INCOMPRESSIBLE BOUNDARY LAYER EQUATIONS WITH NEGATIVE PRESSURE GRADIENT AND LARGE INJECTION RATES

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During entry of bodies into the Earth's atmosphere with high velocities, the mass removal from the body surface as a result of the large convective and primarily radiation fluxes may become arbitrarily large, i.e., the injection rate into the boundary layer may approach infinity. The present article presents a solution of the Prandtl equations for the incompressible boundary layer with negative pressure gradient (dp/dx < 0) for large injection rates. The existence of a solution of the boundary layer equations with arbitrary injection rate under the condition dp/dx < 0 was shown in Oleinik's work [1].

The asymptotic solution obtained agrees with the exact numerical solution for those values of the injection rate for which the boundary layer approximation still remains valid. An analogous solution for the self-similar equations in the vicinity of the stagnation point was previously obtained in [2]. The use of the asymptotic solution makes it possible to find an expression for the friction coefficient which is convenient for concrete calculations in the case of arbitrary negative pressure gradients.

§1. The starting point will be the Prandtl equations

\[ \frac{u}{\partial x} + v \frac{\partial u}{\partial y} = U \frac{\partial U}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2}, \]

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1.1} \]

with the boundary conditions

\[ u(x, 0) = 0, \]

\[ \nu(x, 0) = \nu_0(x), \quad u(x, \infty) = U(x). \tag{1.2} \]

Here \( x \), \( u \) are distance and velocity along the body surface; \( y, v \) are distance and velocity along the normal to the body; \( U \) is the potential flow velocity.

Hereafter it is convenient to change from these variables to new variables, using the transformations suggested in [3]:

\[ \xi = \frac{1}{L V \nu} \int U(x) dx, \quad \eta = \frac{1}{\sqrt{V L \nu}} y. \]

We introduce the streamfunction \( \psi(x, y) \) such that

\[ u = \partial \psi / \partial y, \quad v = - \partial \psi / \partial x, \quad \psi(x, y) = \sqrt{V L \nu} 2 \xi / (\xi, \eta). \]

Here \( V \) and \( L \) are the characteristic velocity and characteristic distance. Then Eqs. (1.1) become

\[ \frac{\partial \psi}{\partial \eta^2} + 2 \xi \frac{\partial \psi}{\partial \eta} + \Lambda \left[ 1 - \left( \frac{\partial \psi}{\partial \eta} \right)^2 \right] = 0, \]

\[ \frac{\partial \psi}{\partial \eta} + \frac{1}{2} \left( \frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \eta^2} \right) = 0. \tag{1.3} \]

Equations (1.2) take the form

\[ f(\xi, 0) = 0, \quad f(\xi, \infty) = 1, \quad f(\xi, 0) = -a (1) \]

\[ \left( \Lambda = \frac{1}{\sqrt{V L \nu}} \frac{U}{2 \xi^2} \right), \quad \nu_0 = \left( \frac{V}{L \nu} \frac{U}{2 \xi^2} \right). \tag{1.4} \]

With \( \xi = 0 \) we obtain the problem for an ordinary differential equation. This problem was solved in [2]. The last of the conditions (1.4) takes precisely this form if we require that as \( \xi \to 0 \) the solution of (1.3) becomes the solution obtained in [2].

The basic problem will be to find the friction coefficient

\[ c_f = \frac{1}{2 \nu U^2} \left( \frac{\partial u}{\partial y} \right)_{y=0} = \left( \frac{V}{L \nu} \right)^{\frac{1}{2}} U^2 \int \psi(\xi, 0). \]

We see that we must find an explicit expression for \( f(\xi, 0) \).

Returning to (1.3), we make the following transformations:

\[ \xi = \eta / \alpha, \quad \lambda = -f / \alpha, \quad \omega(\xi, \lambda) = \left( \frac{\partial \psi}{\partial \xi} \right)^2 \]

Then (1.3) and (1.4) become

\[ \frac{V}{L \nu} - \frac{2 \xi}{\alpha^2} \frac{\partial \omega}{\partial \lambda} + \lambda \frac{\partial \omega}{\partial \xi} - 2 \Delta \omega = 2 \xi \frac{\partial \omega}{\partial \xi}, \tag{1.5} \]

\[ \omega(\xi, 1) = 1, \quad \omega(\xi, -\infty) = 0. \tag{1.6} \]

Equation (1.5) has a singularity at the boundary \( \lambda = 1; \) therefore we shall seek its solution near this boundary separately. First we solve (1.5) near the boundary \( \lambda = 1, \) i.e., using the condition \( \omega(\xi, 1) = 1. \)

We consider the magnitude \( \alpha \) of the injection rate close to infinity; therefore, it is natural to seek the solution in the form of the series

\[ \omega = \omega_0 + \alpha^2 \omega_1 + \alpha^4 \omega_2 + \ldots \tag{1.7} \]

Then (1.5) is written in the form of the system of equations

\[ \lambda \frac{\partial \omega_i}{\partial \lambda} - \frac{2 \xi}{\alpha^2} \frac{\partial \omega_i}{\partial \xi} + 2 \omega_i \lambda - \rho_i(\xi, \lambda) = (i = 0, 1, 2, \ldots) \]

\[ \rho_0 = 0, \quad \rho_i = \sum_{k=0}^{i} A_i \frac{\partial \omega_k}{\partial \xi}, \quad (i > 0), \]

\[ A_0 = \frac{\omega_1}{2 \gamma^1 - \omega_0}, \quad A_1 = \frac{\omega_2}{2 \gamma^1 - \omega_0}, \]

\[ A_i = \frac{\omega_i}{2 \gamma^1 - \omega_0} + \sum_{k=2}^{i} B_{i-k} \frac{2k-3)!}{2k-1!} \frac{1}{1 - \omega_0} (k > 0), \]

\[ B_i = \frac{1}{\gamma_{i-k} \sum_{n=1}^{i} (nk - t + n) \omega_{n+k} B_{i-n}^k} (t > 0). \tag{1.8} \]

We take the boundary conditions for (1.8) in the form

\[ \omega_i(\xi, 1) = 1, \quad \omega_i(\xi, -\infty) = 0 \quad (i > 0). \tag{1.9} \]
The general solution of each equation of (1.8), has the form
\[ \omega_t = \frac{1}{2U^2(\xi)} \left\{ \Phi(\xi^2) + \int_0^\xi \Phi(t) \rho_1(t, t') dt' \right\}. \]

Here we understand \( U(\xi) \) to stand in place of \( x \); the function \( \Phi \) has the same sense.

Using the boundary conditions (1.9), we obtain
\[ \omega_t = \frac{U^2(\xi^2)}{2U^2(\xi)}, \]
\[ \omega_t = \frac{1}{2U^2(\xi)} \int_U U^2(t) \rho_1(t, (\xi^2/t)^1/2) \frac{dt}{t} \quad (t > 0). \quad (1.11) \]

Now we solve (1.5) near the boundary \( \lambda = -\infty \), i.e., we use the condition \( \omega(\xi, -\infty) = 0 \). We shall seek the solution in the form of the series
\[ \Omega(\xi, -\alpha \lambda) = C_1(\alpha) \Omega_1(\xi, -\alpha \lambda) + C_2(\alpha) \Omega_2(\xi, -\alpha \lambda) + \ldots. \quad (1.12) \]

Here
\[ \lim C_1(\alpha) = 0, \quad \lim \frac{C_{m+1}(\alpha)}{C_m(\alpha)} = 0 \quad \text{as} \quad \alpha \to \infty. \quad (1.13) \]

If we consider that the solution is sought in that region where it is close to zero, and if we use the condition (1.13), then it is clear that with sufficiently large values of \( \alpha \) the equation for the function \( \Omega_1 \) has the form
\[ \alpha^2 \Omega_1 / \partial \alpha^2 + i \alpha \partial \Omega_1 / \partial \alpha = -2 \lambda \Omega_1 + \frac{2 \lambda \Omega_1}{\partial \xi} \quad (t = -\alpha \lambda). \quad (1.14) \]

For (1.14) we have the boundary condition
\[ \Omega_1(\xi, \infty) = 0. \quad (1.15) \]

In addition, we must match the solution of (1.14) with the solution (1.11). We shall seek \( \Omega_1(\xi, t) \) in the form
\[ \Omega_1(\xi, t) = P(\xi) T(t). \quad (1.16) \]

Then (1.14) breaks down into two equations:
\[ \xi dP / d\xi + P \lambda = \lambda P, \quad T'' + tT' = tT. \quad (1.17) \]

Assume that the second of these equations has the solution
\[ \gamma = \frac{C_\nu}{U^2(\xi)} \gamma^\nu, \quad \gamma^\nu = \gamma^\nu(\xi, t). \quad (1.18) \]

Using the boundary conditions (1.9), we obtain
\[ \omega_0 = \frac{U^2(\xi^2)}{U^2(\xi)}, \]
\[ \omega_t = \frac{1}{2U^2(\xi)} \int_U U^2(t) \rho_1(t, (\xi^2/t)^1/2) \frac{dt}{t} \quad (t > 0). \quad (1.11) \]

Now we solve (1.5) near the boundary \( \lambda = -\infty \), i.e., we use the condition \( \omega(\xi, -\infty) = 0 \). We shall seek the solution in the form of a series similar to (1.7); we let \( t \to -\infty \), which corresponds to \( \lambda \to 1 \), and we equate the result of the expansion (1.7), letting \( \lambda \to 0 \) in the latter.

Considering that \( \lambda \gg 1 \), we find that as \( t \to -\infty \)
\[ \Omega \to C_1 \frac{e^{2imt/\xi}}{U^2(\xi)} \]
\[ \times \{ \gamma^{2m} \alpha^{m(\lambda^m + m(4m - 1))} \frac{1}{\alpha^{2m^2+...}} \}. \quad (1.22) \]

As \( \lambda \to 0 \) the expansion (1.7) with account for (1.11) takes the form
\[ \omega \to \frac{e^{2imt/\xi}}{U^2(\xi)} \{ \gamma^{2m} + m(4m - 1) \frac{1}{\alpha^{2m^2+...}} \}. \quad (1.23) \]

For agreement of (1.22) and (1.23) it is necessary that \( C_1 = \alpha^{-2m(2\pi)^{-1/2}} \).

Since \( \omega = 1 - (\partial \lambda / \partial z)^2 \), it follows that \( \lambda_{zz} = -1/2 \alpha \lambda' \); therefore
\[ \lambda_{zz} = -a \alpha \lambda''(\xi, 0) = \alpha^{-1} \lambda_{xz}''(\xi, 0) = \alpha^{-1} \lambda_{xx}' \omega(\xi, 1). \quad (2.1) \]

For the derivative \( \omega' \lambda \) on the line \( \lambda = 1 \) we have from (1.7)
\[ \frac{\partial \omega_0}{\partial \alpha} = \frac{\partial \omega_0}{\partial \alpha} + \sum_{i=1}^\infty \frac{1}{\alpha^i} \frac{\partial \omega_0}{\partial \alpha}, \quad (2.2) \]