A simple method of constructing a minimal polynomial basis and a minimal complete system of unitary invariants of a square matrix of the third order is presented.

D. P. Zhelobenko [1] has found a minimal polynomial basis of unitary invariants of a square matrix of the second order. In the present note a simple method is given for constructing a minimal polynomial basis of unitary invariants of a square matrix of the third order.

By a unitary invariant of a variable (3 \times 3)-matrix \( \mathbf{A} \) we mean a polynomial \( f(\mathbf{A}) \) of elements of the matrix \( \mathbf{A} \) and their conjugates, taken over the field of complex numbers, for which

\[
I(\mathbf{VAV}'') = f(\mathbf{A}),
\]

for arbitrary constant (3 \times 3)-matrices \( \mathbf{A} \) and \( \mathbf{U} \), the matrix \( \mathbf{U} \) being unitary.

It is known (cf. [2], Corollary 1) that a polynomial \( f(\mathbf{A}) \) is a unitary invariant of a matrix \( \mathbf{A} \) if and only if it is representable in the form of a polynomial in the traces of the products of powers of the matrices \( \mathbf{A} \) and \( \mathbf{A}^* \), which thus form a polynomial basis of unitary invariants of the matrix \( \mathbf{A} \).

For constructing a minimal polynomial basis we shall use, for two arbitrary (3 \times 3)-matrices \( \mathbf{B} \) and \( \mathbf{C} \), the Cayley-Hamilton equation

\[
\mathbf{B}^3 = \mathbf{B}^2 \mathbf{S} - \frac{1}{2} \mathbf{B} (\mathbf{S}^3 \mathbf{B} - \mathbf{S}^2 \mathbf{B}^2) + \frac{1}{6} \mathbf{E} (\mathbf{S}^3 \mathbf{B} - 3 \mathbf{S} \mathbf{B} \mathbf{B}^2 + 2 \mathbf{S} \mathbf{B}^3)
\]

and its generalization

\[
\mathbf{BCB} = -\mathbf{B}^2 \mathbf{C} + \mathbf{B} (\mathbf{C} + \mathbf{CB}) \mathbf{S} + \mathbf{B}^2 \mathbf{C} + \mathbf{B} (\mathbf{S} \mathbf{B} \mathbf{C} - \mathbf{S} \mathbf{B} \mathbf{C} \mathbf{S} + \frac{1}{2} \mathbf{C} (\mathbf{S} \mathbf{S} \mathbf{S} - \mathbf{S}^2 \mathbf{B}) + \frac{1}{2} \mathbf{S} \mathbf{C} (\mathbf{S}^3 \mathbf{B} - \mathbf{S} \mathbf{S} \mathbf{B}^2))
\]

which is easily obtainable by replacing in Eq. (1) \( \mathbf{B} \) by \( \mathbf{B} + \lambda \mathbf{C} \) and equating to zero the sum of terms involving \( \lambda \) to the first power [3].

Taking note of Eq. (1) and the fact that \( \mathbf{S} \mathbf{B} \mathbf{C} = \mathbf{S} \mathbf{C} \mathbf{B} \), it is sufficient, for the purpose of obtaining the desired basis, to limit consideration, besides the traces

\[
\mathbf{S} \mathbf{A}, \mathbf{S} \mathbf{A}^*, \mathbf{S} \mathbf{A}^2, \mathbf{S} \mathbf{A}^2, \mathbf{S} \mathbf{A}^2, \mathbf{S} \mathbf{A}^2
\]

to traces of products of the form

\[
\mathbf{A}^{\alpha_1} \mathbf{A}^{\alpha_2} \mathbf{A}^{\alpha_3} \mathbf{A}^{\beta_1} \ldots \mathbf{A}^{\beta_k},
\]

where in \( 1 \leq \alpha_r \leq 2, 1 \leq R \leq 2 \) for all \( r = 1, 2, \ldots, k \).

In the set of products of the form (4) we introduce a complete ordering in the following way. We shall say that

\[
\mathbf{A}^{\alpha_1} \mathbf{A}^{\alpha_2} \mathbf{A}^{\alpha_3} \mathbf{A}^{\beta_1} \ldots \mathbf{A}^{\beta_k} < \mathbf{A}^{\alpha_1} \mathbf{A}^{\alpha_2} \mathbf{A}^{\alpha_3} \mathbf{A}^{\beta_1} \ldots \mathbf{A}^{\beta_k},
\]

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if
\[ \sum_{r=1}^{t} (\alpha_r + \beta_r) \leq \sum_{r=1}^{t} (\gamma_r + \delta_r), \]

and, in the case of equality, either \( k < l \) or \( k = l \), but
\[ (\alpha_r - \gamma_r)^2 + (\beta_r - \delta_r)^2 \neq 0 \]

(5)

if only for a single \( r \) (1 \( \leq \) \( r \) \( \leq \) \( k \)), whereby if \( s \) is the first of the numbers 1, 2, ..., \( k \) for which inequality (5) is satisfied, then either \( \alpha_S + \beta_S < \gamma_S + \delta_S \), or \( \alpha_S + \beta_S = \gamma_S + \delta_S \), but \( \gamma_S < \gamma_S \).

We call the product (4) reducible if its trace can be expressed as a polynomial in the traces of smaller products and the traces (3). We shall show that for \( k > 1 \), the only irreducible product is \( A A^* A^2 A^*^2 \).

From formula (2) it follows that an arbitrary product of the form (4) is reducible if only one pair of the numbers \( \alpha_r \) or one pair of the numbers \( \beta_r \) coincides.

If we multiply Eq. (2) on the left by \( B \), we find that products containing \( A^2 \) to the left of \( A \) or \( A^*^2 \) to the left of \( A^* \) are also reducible. From this it then follows that all products of the form (4) with \( k > 1 \) and distinct from \( A A^* A^2 A^*^2 \) are reducible. It is not difficult now to show that the following theorem holds:

**THEOREM.** The polynomials (3), together with the traces
\[ S p A A^* , \quad S p A^2 A^* , \quad S p A A^* , \quad S p A^2 A^*^2 \]
form a minimal polynomial basis of unitary invariants of the \((3 \times 3)\)-matrix \( A \).

It remains only to show that one of the traces (3) and (6) can be expressed as a polynomial in terms of the others. For the polynomials (3), \( S p A^2 A^* \), and \( S p A A^*^2 \), this follows from the fact that no one of the equations
\[
\begin{align*}
S p A^2 &= \alpha S p^2 A , \\
S p A^3 &= \alpha S p A^3 + \beta S p^3 A , \\
S p A^2 A^*^2 &= \alpha S p A A^* S p A + (\beta S p A^* + \gamma S p^2 A) S p A^* \\
\end{align*}
\]
can be satisfied identically for any values of \( \alpha \), \( \beta \), and \( \gamma \). This fact is not needed in a specific proof if examples 5), 6), and 7), introduced below, are considered. That no one of the invariants
\[ S p A A^* , \quad S p A^2 A^*^2 , \quad S p A A^* A^2 A^*^2 \]
(7)
can be expressed as a polynomial, or even in the form of a single-valued function,† of the remaining traces (3) is shown by the following examples:

1) \( A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \)

2) \( A_1 = \begin{pmatrix} 0 & 15 & 0 \\ 0 & 0 & 16 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 20 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \)

3) \( A_1 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \)

In each of these examples two matrices are given for which the values of all the traces (3) and (6), besides one of the traces (7), coincide, respectively.

†From the results of Ya. S. Dubnov (Transactions of a Seminar on Vector and Tensor Analysis, Vol. 5, 1941), it follows that \( S p A A^* A^2 A^*^2 \) satisfies a quadratic equation whose coefficients are polynomials in the remaining ten traces (3) and (6).