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PHASE ANALYSIS IN THE PROBLEM OF SCATTERING BY A RADIAL POTENTIAL

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Let $\delta(k, q)$ be the total scattering cross section of a three-dimensional quantum particle of energy $k^2$ by a radial potential $qV(\rho)$, $\rho = |x|$. Under the assumption $V(\rho) = 0$, $0 < \rho < \rho_0$, it is shown that in the domain $qK \rightarrow \infty$, $q^{3/2}K^{3/2} \rightarrow \infty$ one has the asymptotics $\delta(k, q) \sim \alpha(qK)^{3/2}, \alpha \sim (\alpha - 1)^{3/2}$, where the coefficient $\alpha$ is expressed explicitly in terms of the Gamma function. For nonnegative potentials, the domain of validity of this asymptotic is even larger. For potentials with a strong positive singularity $V(\rho) \sim \rho^{-\delta}$, $\rho > \rho_0$, it is established that $\delta(k, q) \sim q^{3/2}K^{3/2}$ as $qK \rightarrow \infty$. Similar results are obtained for the forward scattering amplitude.

1. Introduction

The scattering cross section (see the definition, for example, in [1]) is one of the fundamental observed quantum mechanics quantities. Let $\delta(k, q)$ be the total scattering cross section, averaged over the incidence directions, of a quantum particle of energy $k^2$ by the potential $qV(\rho)$, $\rho \in \mathbb{R}^3$. We assume that $V(\rho) = 0$ as $|\rho| \rightarrow \infty$ so that the quantity $\delta(k, q)$ is finite. We take the mass of the particle equal to $1/2$ and the Planck constant equal to $1$.

We discuss the behavior of $\delta(k, q)$ for large values of the parameters $k$ and (or) $q$. It is well known (see, for example, [1]) that for $k \rightarrow \infty$ and fixed $q$ (and also for fixed $k$ and $q \rightarrow 0$) the asymptotics of $\delta(k, q)$ is described by the formulas of the theory of perturbation (the Born approximation). These formulas can be rigorously substantiated [2-4] and they hold when both parameters tend to infinity, provided $q/k \rightarrow 0$. Applied to the scattering cross section, the Born approximation shows that for $q/k \rightarrow 0$ we have

$$\delta(k, q) \sim \frac{1}{(8\pi)^{1/2}} \int V(\rho) \, V(q) |x-\rho|^2 \, dq \, d\rho (q/k)^2. \quad (1.1)$$

The behavior of $\delta(k, q)$, outside the framework of perturbation theory, has been investigated recently in [5]. For finite functions $V$, in [5] one has found the (finite) limit of $\delta(k, q)$ as $q \rightarrow \infty$, $k \rightarrow \infty$, $q/k - \text{const}$ (this is equivalent to the fact that the Planck constant tends to zero under fixed values of the remaining parameters of the problem). This result has required certain assumptions on the properties of the corresponding classical system. For functions $V$ with noncompact support, such that $V(x) = 0(|x|^{-2})$, $l > 2$, in [6] (see also the preceding paper [7]) one has obtained the upper estimate $\delta(k, q) \leq C(q/k)^{3/2}$. Strictly speaking, this inequality has been proved in [6] for $k > k_0 > 0$, $q/k > C_0 > 0$ and under a certain averaging of the quantity $\delta(k, q)$ with respect to $k$.


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For spherically symmetric (radial) functions $V(x) = V(r)$, $r = |x|$, with a powerlike asymptotics $V(r) \sim \frac{1}{r^d}$ as $r \to \infty$, in [1] one has given the approximate formula

$$\sigma(k, q) \sim \varphi_d(q |V|/k)^{\frac{2}{d-1}} \tag{1.2}$$

where the coefficient $\varphi_d$ can be expressed explicitly in terms of the Gamma function (see expression (2.7) below) and the number $q^d k^{d-2}$ is sufficiently large. In Secs. 2-4 of the present paper it is elucidated that, in a definite domain of variation of the parameters $k$ and $q$, relation (1.2) can be understood as an asymptotic equality. More exactly, we show that (1.2) is valid if $q/k \to \infty$, $q^d k^{d-2} \to \infty$. For nonnegative functions $V$, the asymptotics (1.2) holds in the larger domain $q/k \to \infty$, $q^d k^{d-2} \to \infty$. In particular, the asymptotics (1.2) is valid under the quasiclassical limiting process $q/k \to \text{const}$, $k \to \infty$. More detailed conditions for the validity of the asymptotics (1.2) are discussed in Sec. 2. We mention the difference in the character of the relations (1.1), (1.2). According to (1.2), the rate of increase of $\sigma(k, q)$ as $q/k \to \infty$ depends on the rate of decrease of the potential at infinity and the asymptotics of $\sigma(k, q)$ is determined only by the asymptotics of $V(r)$. On the contrary, for $q/k \to 0$ the total scattering cross section always decreases as $(q/k)^{d-1}$, while the coefficient of $(q/k)^{d-1}$ depends on the values of $V(x)$ for all $x$.

In the spherically symmetric case, the scattering cross section $\sigma(k, q)$ is expressed (see below equality (2.4)) in terms of the scattering phases $\varphi_d(k, q)$ for the radial Schrödinger equation with the centrifugal term $\ell(l+1)\psi^2$. Following [1], we derive relation (1.2) from the asymptotics of $\varphi_d(k, q)$ for large values of $k$, $q$, and $l$. It is easy to show that for fixed $k$ and $q$ we have

$$\sigma(k, q) \sim \varphi_d(q |V|/k)^{\frac{2}{d-1}} \tag{1.3}$$

where the number $\varphi_d$ is determined by the equality (2.12). For the proof of (1.2) we require the asymptotics (1.3) in a domain where $k \to \infty$, $q \to \infty$ and $l \to \infty$ and the numbers $q^d k^{d-2}$ and $(l+\ell)^{d-1}$ have the same order. On a heuristic level, such an asymptotics follows from the known quasiclassical expression

$$\varphi_d(k, q) \sim \int_{\nu(k, q)} [k^2 - l(l+1)\psi^2 - q^2]^{\frac{1}{2}} \frac{d\nu}{\nu(k, q)} \tag{1.4}$$

where $\nu(k, q)$ is the largest of all the roots of the equations $k^2 - l(l+1)\psi^2 - q^2 = 0$, $k^2 = l(l+1)\psi^2$. In particular, formula (1.4) shows that the asymptotics of $\varphi_d(k, q)$ must be defined only by the classically accessible domain $\psi > \nu(k, q)$. Such an assertion is easily established for nonnegative potentials but, in the general case, for its verification there arises the additional condition $q^d k^{d-2} \to \infty$. Apparently, formula (1.4) can be justified with the aid of the WKB method which, however, requires assumptions on the behavior of $V' \nu$, $V' \psi$ as $\nu \to \infty$. In order to avoid these assumptions, we prove relation (1.3), by-passing (1.4), with the aid of the so-called method of the phase equation (see [8-10]), reducing the radial Schrödinger equation to a nonlinear first-order differential equation.

Relation (1.3) allows us to find also the asymptotics of other physical quantities, expressed in terms of $\varphi_d(k, q)$, similar to the scattering cross section. In addition to $\varphi_d(k, q)$, in the same domain of variation of the parameters $k$ and $q$ we compute (for $l > 3$) the asymptotics of the forward scattering amplitude $\varphi_f(k, q)$. At the absence of spherical symmetry for $V(x)$, the role of the scattering phases is played by the numbers $\varphi_d(k, q)$, where $\varphi_d(k, q)$ are the eigenvalues of the corresponding scattering matrix. For fixed values of $k$, $q$, and $l \to \infty$, an asymptotics of the form (1.3) for such $\varphi_d(k, q)$ has been obtained in [11, 12] under the condition that $V(x) \sim \Phi(x/|x|)|x|^{-d}$, $|x| \to \infty$. On this basis, in [11] one has formulated a hypothesis on the validity for $q/k \to \infty$, $k > 0$ of the asymptotics $\varphi_d(k, q) \sim \varphi_d(q/k)^{d-1}$, where the coefficient $\varphi_d$ depends only on $\Phi$ and $d$. In the case of spherical symmetry, our investigation establishes the validity of this assumption, although in a somewhat different domain of variation of the parameters $k$ and $q$. Without the assumption of spherical symmetry, the hypothesis from [11]