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Arakelov's Theorem for Abelian Varieties

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§ 1. Introduction

We start with an algebraic curve B over an algebraically closed field k of characteristic zero, and let $S \subseteq B$ be a finite set of points. In [1] Arakelov has shown that there are only finitely many families of algebraic curves of genus $g > 1$ on B, with good reduction outside S, except for isotrivial families (isotrivial = becomes constant on a finite cover of B).

We want to consider the same question for principally polarized abelian varieties. Here the answer is more complicated:

There is a condition $(*)$ such that the number of abelian varieties fulfilling $(*)$ is finite, while any variety not fulfilling it can be deformed. The condition $(*)$ says essentially that all endomorphisms of the cohomology of the abelian variety are endomorphisms of the abelian variety itself.

The method of proof consists of a combination of Arakelov's methods and Deligne's description of abelian varieties via Hodge-structures. In the next chapter we recall the necessary prerequisites, and after that we prove the theorem in two steps as in [1].

We first derive a boundedness-theorem and then we show that families fulfilling $(*)$ cannot be deformed. From the form of the theorem it seems that it is difficult to take it over to characteristic $p > 0$ (see [7]). The author has learned about this subject from L. Szpiro, who kindly printed out to him that the following results are already known:

a) L. Moret-Bailly has proved a boundedness-theorem in any characteristic. The result is contained in his thesis. (It will appear in the proceedings of the "seminar on pencils of abelian varieties, Paris 1981/82", and in the Comptes-Rendues). Unfortunately this theorem is weaker in characteristic zero than ours.

b) L. Szpiro and L. Moret-Bailly have a very good theorem about boundedness for $S = \emptyset$ (any characteristic).

c) They have derived rigidity for relative dimension two. (To appear also in "seminar on pencils of abelian varieties"). In characteristic zero our results cover relative dimension up to three.

The results were found during a stay at the I.H.E.S. I have to thank P. Deligne for some help concerning the example of an abelian variety not fulfilling (*).

The referee has told me that the results of Zucker (Ann. of Math. **109**) should allow to treat more general Hodge-structures on $B-S$. As *I* am not an expert in this field *I* leave this to the reader. In any case *I* thank the referee for his suggestions.

§ 2. Notations

k always denotes an algebraically closed field of characteristic 0, and in the proofs we assume that $k=\mathbb{C}$, which we may do by the Lefschetz-principle. B denotes a connected complete smooth curve over k, and $S\subseteq B$ a finite set of points. We want to consider families

$$p\colon X\to B-S$$

of abelian varieties of relative dimension g over S. By [4] we know that giving such a family is the same as giving a polarizable Hodge-structure (or variation of Hodge-structure) of degree 1 on $B-S$, that is (except for the polarization) a locally constant sheaf $\mathbf{V}$ on $B-S$, locally isomorphic to $\mathbb{Z}^{2g}$, plus a subbundle

$$\mathscr{W}=\mathscr{W}_X\subseteq \mathbf{V}\otimes_{\mathbb{Z}}\mathcal{O}_B$$

of rank g, such that $\mathscr{W}$ and its complex conjugate span the fibre of $\mathbf{V}$ in each point. We furthermore assume that it has a principal polarization, which means a skew-symmetric form

$$\langle\ ,\ \rangle\colon \mathbf{V}\times\mathbf{V}\to\mathbb{Z},$$

which identifies $\mathbf{V}$ with its own dual, vanishes on $\mathscr{W}\times\mathscr{W}$, and which has the additional property that for $w\in\mathscr{W}$ and $w\neq 0$

$$-\frac{1}{2\pi i}\langle w,\bar{w}\rangle>0.$$

Definition. (See [4], § 4.4.) A family

$$p\colon X\to B-S$$

satisfies (*) if any anti-symmetric endomorphism A of

$$\mathbf{V}=R^1p_*\mathbb{Z}$$

defines an endomorphism of X (i.e., is of type $(0,0)$). A is called anti-symmetric if

$$\langle \mathrm{A}u,v\rangle=-\langle u,\mathrm{A}v\rangle.$$

(We may use étale or de Rham cohomology to formulate that over base fields $k\neq\mathbb{C}$.)