

Heegner points and derivatives of L -series

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to John Tate

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I. Introduction and statement of results

The main theorem of this paper gives a relation between the heights of Heegner divisor classes on the Jacobian of the modular curve $X_0(N)$ and the first derivatives at $s=1$ of the Rankin L -series of certain modular forms. In the first six sections of this chapter, we will develop enough background material on modular curves, Heegner points, heights, and L -functions to be able to state one version of this identity precisely. In §7 we will discuss some applications to the conjecture of Birch and Swinnerton-Dyer for elliptic curves. For example, we will show that any modular elliptic curve over \mathbb{Q} whose L -function has a simple zero at $s=1$ contains rational points of infinite order. Combining our work with that of Goldfeld [12], one obtains an effective lower bound for the class numbers of imaginary quadratic fields as a function of their discriminants (§8). In §9 we will describe the plan of proof and the contents of the remaining chapters.

Many of the results of this paper were announced in our Comptes Rendus note [17]. A more leisurely introduction to Heegner points and Rankin L -series may be found in our earlier paper [13].

§1. The curve $X_0(N)$ over \mathbb{Q}

Let $N \geq 1$ be an integer. The curve $X = X_0(N)$ may be informally described over \mathbb{Q} as the compactification of the space of moduli of elliptic curves with a cyclic subgroup of order N . It is known to be a complete, non-singular, geometrically connected curve over \mathbb{Q} . Over a field k of characteristic zero, the points x of X correspond to diagrams

$$(1.1) \quad \phi: E \rightarrow E'$$

where E and E' are (generalized) elliptic curves over k and ϕ is an isogeny over k whose kernel A is isomorphic to $\mathbb{Z}/N\mathbb{Z}$ over an algebraic closure \bar{k} . The function field of X over \mathbb{Q} is generated by the modular invariants $j(x) = j(E)$ and $j'(x) = j(E')$; these satisfy the classical modular equation of level N : $\phi_N(j, j') = 0$ [2].

The cusps of X are the points where $j(x) = j'(x) = \infty$. They correspond to diagrams (1.1) between certain degenerate elliptic curves, where $A = \ker \phi$ meets each geometric component of E [7, 173ff.]. There is a unique cusp where E has 1 component and a unique cusp where E has N components; these are denoted ∞ and 0 respectively and are rational over \mathbb{Q} .