Extrapolation Methods for Dynamic Partial Differential Equations

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Summary. Several extrapolation procedures are presented for increasing the order of accuracy in time for evolutionary partial differential equations. These formulas are based on finite difference schemes in both the spatial and temporal directions. One of these schemes reduces to a Runge-Kutta type formula when the equations are linear. On practical grounds the methods are restricted to schemes that are fourth order in time and either second, fourth or sixth order in space. For hyperbolic problems the second order in space methods are not useful while the fourth order methods offer no advantage over the Kreiss-Oliger method unless very fine meshes are used. Advantages are first achieved using sixth order methods in space coupled with fourth order accuracy in time. The averaging procedure advocated by Gragg does not increase the efficiency of the scheme. For parabolic problems severe stability restrictions are encountered that limit the applicability to problems with large cell Reynolds number. Computational results are presented confirming the analytic discussions.

Subject Classifications. AMS (MOS): 65M99; CR: 5.17.

I. Introduction

In recent years extrapolation to the limit has become a popular technique for solving ordinary differential equations (Gragg [8]; Bulirsch and Stoer [2]). In applying this idea to partial differential equations one must decide whether the extrapolation is to be done for all independent variables or only in time. An

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extrapolation method in space and time has been described by Smith and McCall [19] and Busch et al. [3] based on the method of characteristics. However, this method is applicable only to two equations in one space dimension plus time and so is of limited use. For standard finite difference methods it would be difficult to account for the boundary conditions while verifying the appropriate expansions in both space and time. Furthermore, space refinement of the grid is a very expensive procedure both in computer storage and computer running time, especially for multidimensional problems. The minimum mesh widths are determined by features of the boundary, the need to describe small viscous effects (see Cheng [4]) and the ability to follow high frequency waves (see Kreiss and Oliger [11]). Hence, in many problems it is not feasible to change the spatial grid to any significant degree. In addition Skollemo [18] has shown that in some cases extrapolation is valid only for the even mesh points in both the coarse and fine grids which makes space extrapolation even more expensive. Thus, one is limited to the use of time extrapolations in order to achieve higher accuracy.

As with ordinary differential equations one can perform either local or global extrapolations. For problems that involve wave motion one cannot use global time extrapolations unless the solution given by the larger time steps is already very accurate. This is because of the appearance of phase errors in the numerical solution. After long time integrations the numerical solution lags (or leads) the true solution and the amount of lag depends on the size of the time step. Use of extrapolation at a fixed time level will obviously not lead to any improvement in the solution. One should perform the extrapolation along characteristic directions; however this is difficult in practice especially for multidimensional systems of equations. Another difficulty with global extrapolations is imposed by stability limits. This prevents us from using larger time steps but instead requires the use of finer time steps. The use of the very small time steps again greatly increases the running time of the program. We therefore conclude that for practical problems one is limited to local time extrapolations. A more complete discussion for ordinary differential equations is given by Lapidus and Seinfeld [12].

There is an additional difficulty which further limits the applicability of extrapolation methods to partial differential equations. It has been observed (Oliger [14]; Gary [5]) that finite difference formulas are most efficient when the errors in space and time are about equal. The orders of accuracy in space are usually two, four or possibly six. It would therefore not be efficient to extrapolate, in time, to the limit. Hence, we shall only consider extrapolation methods that lead to fourth order methods in time. These are coupled with second, fourth and sixth order methods in space.

A possible alternative is to use spectral methods in the space variables which yield infinite order accurate spatial solutions (Orszag [15]). It can then be shown that the use of lower order methods in time dominates the solution and prevents the effective use of the higher order spatial resolution. For these methods extrapolation to the limit in the temporal direction is of importance. However, in this study we shall confine our attention to finite difference formulas.