Newton's Method for Gradient Equations Based upon the Fixed Point Map: Convergence and Complexity Study

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Summary. An approximate Newton method, based upon the fixed point map \( T \), is introduced for scalar gradient equations. Although the exact Newton method coincides for such scalar equations with the standard iteration, the structure of the fixed point map provides a way of defining an \( R \)-quadratically convergent finite element iteration in the spirit of the Kantorovich theory. The loss of derivatives phenomenon, typically experienced in approximate Newton methods, is thereby avoided. It is found that two grid parameters are essential, \( h \) and \( h \approx h^2 \). The latter is used to calculate the approximate residual, and is isolated as a fractional step; it is equivalent to the approximation of \( T \). The former is used to calculate the Newton increment, and this is equivalent to the approximation of \( T' \). The complexity of the finite element computation for the Newton increment is shown to be of optimal order, via the Vituškin inequality relating metric entropy and \( n \)-widths.

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Introduction

In this paper, we introduce a nonstandard approximate Newton method for the gradient equation,

\[- \nabla \cdot [a(x) \nabla u(x)] + f(x, u(x)) = g(x), \quad (1.1)\]

on an open, convex polyhedral domain \( \Omega \subset \mathbb{R}^d \), subject to the Dirichlet boundary condition \( u = \bar{u} \) on \( \partial \Omega \), where it is assumed that

- \( a \in \text{Lip}(\Omega) \) and \( 0 < a_0 \leq a(x) \leq a_1 \),
- \( a_1 \leq f(x, u) \leq a_2 \), for all \( x \in \Omega, u \in \mathbb{R} \),
- \( 0 < \beta_1 \leq f_u(x, u) \leq \beta_2 \), for all \( x \in \Omega, u \in \mathbb{R} \),
- the extension of \( \bar{u} \) to \( \Omega \) satisfies \( \bar{u} \in H^2(\Omega) \cap L^\infty(\Omega) \),

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\* \( g \in L^\infty(\Omega) \),
\* \( f_u(x, \cdot) \) is Lipschitz continuous on \( \mathbb{R} \), with \( \| f_u(x, \cdot) \|_{\text{Lip}} \leq c \), for all \( x \in \Omega \).  

(1.2)

Thus, \( f(x, \cdot) \) is a monotonically increasing invertible function for each \( x \in \Omega \). The framework permits functions such as

\[
f(x, \, u) = e^u,
\]

(1.3)

even though such an \( f \) is not bounded in \( u \), because of the existence of maximum principles for (1.1). Thus we have,

\[
\gamma \leq u \leq \delta,
\]

(1.4)

where \( \gamma \) and \( \delta \) are defined through the intermediate

\[
\gamma' = \inf_{x \in \Omega} f^{-1}(x, g(x)) \quad \text{and} \quad \delta = \sup_{x \in \Omega} f^{-1}(x, g(x))
\]

by

\[
\gamma = \min \left[ \gamma', \inf_{\partial \Omega} \tilde{u} \right] \quad \text{and} \quad \delta = \max \left[ \delta', \sup_{\partial \Omega} \tilde{u} \right].
\]

In particular, if \( f \) is defined by (1.3), the bounds (1.4) make possible an extension \( f \) of \( \exp(\cdot) \), from the interval \([\gamma, \delta]\) to \( \mathbb{R} \), such that \( f \) and \( f_u \) are bounded.

The approximate Newton method introduced here is based upon the fixed point map, \( u \mapsto Tu \), where

\[
- \nabla \cdot [a(x) \nabla Tu(x)] + f(x, u(x)) = g(x), \quad (1.5a)
\]

\[
[Tu - \tilde{u}]_{\partial \Omega} = 0. \quad (1.5b)
\]

In terms of this map, to be described precisely in Sect. 2, the new exact Newton iterate \( w \) is obtained from the previous iterate \( v \) by the solution of the Dirichlet problem,

\[
- \nabla \cdot (a \nabla w) + \frac{\partial f}{\partial u} (\cdot, v) w = g - f(\cdot, v) + \frac{\partial f}{\partial u} (\cdot, v) v, \quad (1.6a)
\]

\[
(w - \tilde{u})|_{\partial \Omega} = 0. \quad (1.6b)
\]

What is noteworthy is the absence of explicit references to \( T \) and \( T' \) from the formula (1.6), in spite of the operator characterization of \( w \), via

\[
w = T'(v)(w - v) + Tv. \quad (1.7)
\]

The reader may verify that exact standard Newton iteration, based directly upon (1.1), leads to the same characterization (1.6) in this case of a single scalar equation; however, approximate Newton methods, based upon (1.6), are not in theory effective, whereas those based upon (1.7) are. We elaborate the reasons for this in the next paragraph.