P-Stability Properties of Runge-Kutta Methods for Delay Differential Equations*

Marino Zennaro
Dipartimento di Scienze Matematiche Università degli Studi di Trieste, 1-34100 Trieste, Italy

Summary. Recently the author and others have studied a class of Runge-Kutta methods for DDEs. These methods make use of certain constrained meshes, which allow to get optimal order results. In this paper we study their asymptotic stability properties, using the test equation

\[ y'(t) = ay(t) + by(t - \tau), \quad t > 0 \]
\[ y(t) = g(t) \quad \text{for } -\tau \leq t \leq 0 \]

where \( a, b \in \mathbb{C}, \tau > 0, \) and \( g(t) \) is continuous and complex valued. It is known that \( y(t) \to 0 \) as \( t \to +\infty \) if \( |b| < -\text{Re}(a) \). In particular, we show that any A-stable one-step collocation method for ODEs inherits the same property when it is applied to DDEs with a constrained mesh (i.e. it is P-stable).

Subject Classifications: AMS(MOS): 65L20; CR: G1.7.

1. Introduction

Recently Bellen [3, 4] and [5], Vermiglio [13] and Zennaro [14] have studied a class of methods for the numerical solution of DDEs such as

\[ y'(t) = f(t, y(t), y(t - \tau(t))), \quad t > t_0 \]
\[ y(t) = g(t) \quad \text{for } t \leq t_0 \] (1)

where \( y, f \) and \( g \) are \( n \)-vectors, \( \tau(t) > 0 \) and \( \phi(t) := t - \tau(t) \) is an invertible function.

A common feature of these methods is the use of a constrained mesh. In order to define a constrained mesh for the DDE (1), we have to know the so-called breaking points of the solution \( y \), that is the points at which \( y \) has discontinuities in its derivatives, caused by the functional argument \( \phi(t) \). These points are \( \xi_0 = t_0 < \xi_1 < \ldots < \xi_j < \ldots, \xi_{j+1} \) solution of \( \phi(\xi_j) = \xi_j \).

* This work was supported by the Italian M.P.I. (Ministero della Pubblica Istruzione)
Definition 1. For \( j \geq 1 \), put \( I_j = [\xi_{j-1}, \xi_j] \). Then, a \textit{constrained mesh} is any mesh \( \Delta \) satisfying the following properties:

(i) it includes the breaking points;
(ii) it is arbitrarily chosen in \( I_1 \);
(iii) it is such that \( \Delta \cap I_j = \phi^{-1}(\Delta \cap I_{j-1}) \) for \( j \geq 2 \).

Bellen [3] has proved that, by using this kind of meshes, the one-step collocation method at \( v \) Gaussian points retains the superconvergence order \( 2v \) at the nodes, although the collocation solution is used to approximate the retarded part, which is uniformly accurate of order \( v + 1 \) only. Later Vermiglio [13] obtained the same results for another implicit \( v \)-level Runge-Kutta method of order \( 2v - 2 \). Finally, Zennaro [14] defined the NCEs (Natural Continuous Extensions) of all Runge-Kutta processes and proved that, if they are used to approximate the retarded part, then, for constrained meshes, the nodal order \( p \) of the Runge-Kutta method is preserved. This happens although the order of uniform accuracy of the NCEs can be, in general, lower than \( p \). Remark that \( p \) is a sufficient order of uniform accuracy to guarantee the maintenance of the nodal order \( p \) for all meshes (see, for example, Oberle-Pesch [10]).

Definition 2. We shall call \textit{R-K method for DDEs} any R-K (Runge-Kutta) process applied to DDEs such as (1) with a constrained mesh and with the use of a NCE for the approximation of the retarded part.

Remark that the methods in [3] and in [13] belong to this class (see [14]).

In this paper we find some stability properties of the R-K methods for DDEs. There are many concepts of stability for DDEs, which are based on different test equations (e.g. Al-Mutib [1], Barwell [2], Cryer [6], Jackiewicz [8], Jackiewicz-Bakke [9], van der Houven-Sommeijer [12]). We restrict our attention to equations such as (1) in which the delay \( \tau \) is constant and we follow the approach of Barwell [2]. It is based on the linear test equation

\[
\begin{align*}
y'(t) &= ay(t) + by(t - \tau), \quad t > 0 \\
y(t) &= g(t) \quad \text{for } -\tau \leq t \leq 0
\end{align*}
\]  

(2)

where \( a, b \in \mathbb{C}, \tau > 0 \) and \( g(t) \) is continuous and complex valued.

It is known that the solution \( y \) of (2) vanishes as \( t \to +\infty \), all delays \( \tau \) and all initial functions \( g \), if

\[
|b| < -\text{Re}(a).
\]  

(3)

We shall see that the weak inequality \((\leq)\) is necessary, too.

Barwell's definitions are the following.

Definition 3. Given a numerical method for DDEs, the \textit{P-stability region} of the method is the set \( S_p \) of the pairs of complex numbers \((\alpha, \beta)\), \( \alpha := ah, \beta := bh \), such that the numerical solution of (2) asymptotically vanishes for step-lengths \( h \) satisfying

\[
h = \tau/m, \quad m \text{ positive integer}.
\]  

(4)

Definition 4. A numerical method for DDEs is said to be \textit{P-stable} if

\[
S_p \supseteq \{ (\alpha, \beta) \in \mathbb{C} \times \mathbb{C} \mid \text{Re}(\alpha) < 0 \quad \text{and} \quad |\beta| < -\text{Re}(\alpha) \}.
\]