

# Polarization and Unitary Representations of Solvable Lie Groups

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## 0. Introduction

If  $G$  is a Lie group (not necessarily connected)  $\hat{G}$  denotes the set of equivalence classes of irreducible unitary representations of  $G$ .

Our principal concern here is for the case when  $G$  is connected, simply connected and solvable. Our main results (1) give a simple, necessary and sufficient condition for  $G$  to be of Type I, (2) determine  $\hat{G}$

in case  $G$  is of Type I and (3) construct elements of  $\hat{G}$  which, in the Type I case, exhaust  $\hat{G}$ . The results extend Kirillov's results for the nilpotent case to Type I solvable groups. These results were announced in [1].

Let  $G$  be a Lie group. Let  $\mathfrak{g}$  be the Lie algebra of  $G$  and let  $\mathfrak{g}'$  be the real dual to  $\mathfrak{g}$ . We regard  $\mathfrak{g}'$  as a  $G$ -module with respect to the coadjoint representation.

If  $G$  is connected, simply connected and nilpotent, then Kirillov has shown that  $\hat{G}$  is, in a natural way, in a one-one correspondence with the set,  $\mathfrak{g}'/G$ , of all orbits of  $G$  operating in  $\mathfrak{g}'$ . Moreover given  $g \in \mathfrak{g}'$ , then the element  $\pi_0 \in \hat{G}$  corresponding to the orbit  $0 = G \cdot g$  was explicitly constructed as an induced representation,  $\text{ind}_G \xi$ , where  $\xi$  is a character (a one dimensional representation) of a subgroup  $H \subseteq G$  whose Lie algebra  $\mathfrak{h}$  is maximal with the property that  $g$  vanishes on the commutator  $[\mathfrak{h}, \mathfrak{h}]$  of  $\mathfrak{h}$ . The character  $\xi$  on  $H$  is the unique such character whose differential is the restriction  $2\pi i g|_{\mathfrak{h}}$ . One of the many remarkable aspects of the Kirillov theorem (see [7]) is that  $\pi_0 = \text{ind}_G \xi$  is independent of the choice of  $H$ . (This of course enables one to associate  $\pi_0$  with the orbit  $0$ .) For example, if  $G$  is the Heisenberg group and  $g$  does not vanish on  $[g, g]$  then the (unique) equivalence for two choices of  $H$  is provided by the Fourier transform.

Although we shall not go into the matter here, Kostant [8] has developed a general theory (a quantization theory) of obtaining unitary representations of an arbitrary Lie group  $G$  from symplectic manifolds on which  $G$  operates. One of the ingredients of the theory is the notion of a polarization of the symplectic manifold. The theory applies to the orbits of  $G$  in  $\mathfrak{g}'$ . If  $0 = G \cdot g$  then a  $G$ -invariant polarization of  $0$  is provided by a polarization at  $g$ . The latter *does* play a main role here so we shall go into more detail.

By a polarization at  $g \in \mathfrak{g}'$  we mean a subalgebra  $\mathfrak{h}$  of the complexification  $\mathfrak{g}_{\mathbb{C}}$  of  $\mathfrak{g}$  such that (1)  $\mathfrak{h}$  is maximal such that  $g$  vanishes on  $[\mathfrak{h}, \mathfrak{h}]$ , (2)  $\mathfrak{h} + \bar{\mathfrak{h}}$  is a subalgebra of  $\mathfrak{g}_{\mathbb{C}}$  and, (3)  $\mathfrak{h}$  is stable under  $\text{Ad } G_g$  where  $G_g$  is the isotropy at  $g$ . This concept is discussed in detail in Sections I.1, I.4 and II.2 below.

The polarization  $\mathfrak{h}$  at  $g$  is said to be defined over  $\mathbf{R}$  in case  $\mathfrak{h} = \bar{\mathfrak{h}}$ . In the general case one defines, in a natural way, a real non-singular symmetric bilinear form  $S_g$  on  $\mathfrak{e}/\mathfrak{d}$  where  $\mathfrak{e} = (\mathfrak{h} + \bar{\mathfrak{h}}) \cap \mathfrak{g}$  and  $\mathfrak{d} = \mathfrak{h} \cap \mathfrak{g}$ . The polarization  $\mathfrak{h}$  is called positive if  $S_g$  is positive definite. If  $\mathfrak{h}$  is defined over  $\mathbf{R}$  it is trivially positive.

The linear functional  $g$  is called integral (with respect to  $G$ ) if there exists a character  $\eta_g$  on  $G_g$  (which may not be connected even if  $G$  is connected and simply connected) whose differential is the restriction of  $2\pi i g$  to the Lie algebra of  $G_g$ . Such a character is called a character at  $g$ .