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Polarization and Unitary Representations of Solvable Lie Groups

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0. Introduction

If G is a Lie group (not necessarily connected) $\hat{G}$ denotes the set of equivalence classes of irreducible unitary representations of G.

Our principal concern here is for the case when G is connected, simply connected and solvable. Our main results (1) give a simple, necessary and sufficient condition for G to be of Type I, (2) determine $\hat{G}$

in case G is of Type I and (3) construct elements of $\hat{G}$ which, in the Type I case, exhaust $\hat{G}$. The results extend Kirillov's results for the nilpotent case to Type I solvable groups. These results were announced in [1].

Let G be a Lie group. Let $\mathfrak{g}$ be the Lie algebra of G and let $\mathfrak{g}'$ be the real dual to $\mathfrak{g}$. We regard $\mathfrak{g}'$ as a G-module with respect to the coadjoint representation.

If G is connected, simply connected and nilpotent, then Kirillov has shown that $\hat{G}$ is, in a natural way, in a one-one correspondence with the set, $\mathfrak{g}'/G$, of all orbits of G operating in $\mathfrak{g}'$. Moreover given $g \in \mathfrak{g}'$, then the element $\pi_0 \in \hat{G}$ corresponding to the orbit $0 = G \cdot g$ was explicitly constructed as an induced representation, $\operatorname{ind}_G \xi$, where ξ is a character (a one dimensional representation) of a subgroup $H \subseteq G$ whose Lie algebra $\mathfrak{h}$ is maximal with the property that g vanishes on the commutator $[\mathfrak{h}, \mathfrak{h}]$ of $\mathfrak{h}$. The character ξ on H is the unique such character whose differential is the restriction $2\pi i g|\mathfrak{h}$. One of the many remarkable aspects of the Kirillov theorem (see [7]) is that $\pi_0 = \operatorname{ind}_g \xi$ is independent of the choice of H. (This of course enables one to associate π_0 with the orbit 0.) For example, if G is the Heisenberg group and g does not vanish on $[\mathfrak{g}, \mathfrak{g}]$ then the (unique) equivalence for two choices of H is provided by the Fourier transform.

Although we shall not go into the matter here, Kostant [8] has developed a general theory (a quantization theory) of obtaining unitary representations of an arbitrary Lie group G from symplectic manifolds on which G operates. One of the ingredients of the theory is the notion of a polarization of the symplectic manifold. The theory applies to the orbits of G in $\mathfrak{g}'$. If $0 = G \cdot g$ then a G-invariant polarization of 0 is provided by a polarization at g. The latter *does* play a main role here so we shall go into more detail.

By a polarization at $g \in \mathfrak{g}'$ we mean a subalgebra $\mathfrak{h}$ of the complexification $\mathfrak{g}_{\mathbf{C}}$ of $\mathfrak{g}$ such that (1) $\mathfrak{h}$ is maximal such that g vanishes on $[\mathfrak{h}, \mathfrak{h}]$, (2) $\mathfrak{h} + \bar{\mathfrak{h}}$ is a subalgebra of $\mathfrak{g}_{\mathbf{C}}$ and, (3) $\mathfrak{h}$ is stable under $\operatorname{Ad} G_g$ where G_g is the isotropy at g. This concept is discussed in detail in Sections I.1, I.4 and II.2 below.

The polarization $\mathfrak{h}$ at g is said to be defined over $\mathbf{R}$ in case $\mathfrak{h} = \bar{\mathfrak{h}}$. In the general case one defines, in a natural way, a real non-singular symmetric bilinear form S_g on $\mathfrak{e}/\mathfrak{d}$ where $\mathfrak{e} = (\mathfrak{h} + \bar{\mathfrak{h}}) \cap \mathfrak{g}$ and $\mathfrak{d} = \mathfrak{h} \cap \mathfrak{g}$. The polarization $\mathfrak{h}$ is called positive if S_g is positive definite. If $\mathfrak{h}$ is defined over $\mathbf{R}$ it is trivially positive.

The linear functional g is called integral (with respect to G) if there exists a character η_g on G_g (which may not be connected even if G is connected and simply connected) whose differential is the restriction of $2\pi i g$ to the Lie algebra of G_g. Such a character is called a character at g.