Inventiones math. 29, 245–274 (1975)

Real Homotopy Theory of Kähler Manifolds

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This paper brings together the strengthened de Rham theory of the last author [12, 13] and [14] concerning the homotopy information of a manifold contained in the algebra of its differential forms, and the classical results [15] about the forms on a compact Kähler manifold to prove statements about the algebraic topology of compact Kähler manifolds, and consequently about smooth, complex, projective algebraic varieties.

The nature of the results is that the *homotopy type of a compact Kähler manifold, over the real numbers, is a formal consequence of the real cohomology ring.* For any manifold there is a differential graded algebra with rational structure constants whose isomorphism type is an invariant of the homotopy type of the space. The cohomology ring of this differential algebra is the rational cohomology ring of the space, but the algebra itself contains, for example, the higher order cohomology products. The isomorphism type of the differential algebra is equivalent to the "rational form of the nilpotent structure in the space". For a simply connected manifold this differential algebra is equivalent to the rational Postnikov tower of manifold. Our results here are that, for a compact Kähler manifold, from the real cohomology ring one can produce the real form of this differential algebra. These results complement the known structure of the cohomology ring of such a manifold. We prove no new results about it.

The initial motivation for these results was the relation in a Kähler manifold between harmonic forms and holomorphic forms. In particular the $(p, 0)$ harmonic forms for any Kähler metric are exactly the holomorphic forms. Consequently, products of $(p, 0)$ and $(q, 0)$ harmonic forms are harmonic[1], and thus there are no

* This research was partially supported by NSF grant GP-38886.

[1] Chen in [2] was the first to exploit this idea to obtain higher order information about Kähler manifolds.

higher order products starting from $H^{*,0}$. The possibility that the full theorem would be true is related to the Weil conjectures. We were lad to prove such a complete result by "using the Weil conjectures in characteristic p to guess results in characteristic 0 which can then be proved using Hodge theory". The relevant remark here is that the classical higher order cohomology products of Massey are multilinear operations, but the dimension of the product is always less than the sum of the dimensions of the various elements being multiplied. Thus, if we were working in a situation in characteristic p on which the Frobenius automorphism operated (for instance, the Čech cochains in étale cohomology theory), and if the eigenvalues of the Frobenius action are as predicted by the Weil conjectures[2], then the multilinearity of the product over the action would imply that the absolute value of the eigenvalues on the space of higher order products in a certain dimension would be different from the possible values in that dimension. The only way this can happen is for these spaces of higher order products to be zero.

In proving these results for compact Kähler manifolds we use the implications for the forms of the existence of a Kähler metric. If M is a complex manifold, form the differential algebras and algebra maps

$$\{H^*(M,\mathbb{R}), d=0\} \leftarrow \{\mathrm{Ker}(d^c), d\} \hookrightarrow \{\text{all } C^\infty \text{ forms}, d\}.$$

If M is a compact Kähler manifold, both maps above induce isomorphisms of cohomology. All our results about the homotopy type of compact Kähler manifolds can be deduced from this statement, which is itself a consequence of the dd^c lemma.

dd^c Lemma. *Let M be a compact Kähler manifold, and $d^c = J^{-1} dJ$, where J gives the complex structure in the cotangent bundle. Evidently, d^c is a real operator. If x is a form with*

1) $dx = 0 = d^c x$

and

2) $x = dy$ or $x = d^c y'$

then $x = dd^c z$ for some form z.

In Sections 1–4 we describe the machinery needed to deduce the consequences in homotopy theory of these statements about forms. Sections 1, 2, and 3 constitute a sketch of the theory in [12] and [14], (c.f. also [10]). Section 1 describes the construction of the minimal model (real Postnikov tower) of any differential algebra and proves a uniqueness result. To do this, some of the abstract homotopy theory of differential algebras is developed. For instance, we define the de Rham fundamental group of an algebra by using a minimal model for the algebra. This "group" is an inverse system of nilpotent Lie groups. We also define the higher homotopy groups of any simply connected differential algebra. These are vector spaces with a graded Lie algebra structure which should be thought of as the structure of "Whitehead products".

In Section 2 we assign to any simply connected[3], C^∞-manifold, M, the de Rham complex of C^∞-forms on M, $\mathscr{E}^*_M$, and its minimal model $\mathscr{M}_M$; and to any simply

[2] Which is now [5].

[3] Actually this form of theory works just as well for nilpotent spaces, see [13].