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Universal Connections

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Let H be a compact Lie group. It is well-known [5] that principal H-bundles over a manifold M are classified by homotopy classes of maps into a certain space BH called the classifying space. In particular, every principal H-bundle over M is the pull-back of the canonical bundle over BH by some map from M into BH. In [4], Narasimhan and Ramanan showed that the canonical bundle over BH has a connection which is universal in the sense that any connection on any principal H-boundle over M may be induced by some map from M to BH.

A simple proof of the Narasimhan-Ramanan theorem will be given in this paper. The method yields the following generalizations: If the connection on the H-bundle is real-analytic, then the map into BH may be chosen to be real-analytic. If the connection is invariant under some compact Lie group G, then the map into BH may be chosen to be equivariant with respect to G. Furthermore, these maps are unique up to a connection preserving homotopy.

§1. The Orthogonal Group

The classifying space for $O(k)$ is the direct limit (as n tends to infinity) of the Grassmanian manifolds $G_k(\mathbb{R}^n) = O(n)/O(k) \times O(n-k)$. The total space of the canonical principal $O(k)$-bundle on $G_k(\mathbb{R}^n)$ is the Stiefel manifold $O(n)/O(n-k)$. The left-invariant (Maurer-Cartan) form $A^{-1}\,dA$ on $O(n)$ determines a k-by-k submatrix of 1-forms ω which is well-defined on $O(n)/O(n-k)$ and is called the canonical connection (see [2]). This connection is compatible with the direct limit.

This can be described more geometrically in terms of the associated vector bundle $\pi\colon \gamma^k(\mathbb{R}^n) \to G_k(\mathbb{R}^n)$. We have

$$G_k(\mathbb{R}^n) = \{W \subset \mathbb{R}^n \mid W \text{ is a subspace of dimension } k\},$$
$$\gamma^k(\mathbb{R}^n) = \{(v, W) \mid v \in W \quad \text{and} \quad W \in G_k(\mathbb{R}^n)\}$$

with $\pi(v, W) = W$. The horizontal curves through (v, W) are given by "rotations of

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v perpendicular to W," that is

$$t \mapsto (\exp(tB)\,v, \exp(tB)\,W)$$

where B belongs to the subspace of $o(n)$ that can be identified with Hom $(W, W^{\perp})$.

The proof of the Narasimhan-Ramanan theorem is motivated by the following well-known

Proposition. *Given a submanifold M^k of $\mathbb{R}^n$, the connection on M induced from $\mathbb{R}^n$ is the same as the one pulled back from $\gamma^k(\mathbb{R}^n)$ via the Gauss map $M \to G_k(\mathbb{R}^n)$.*

Proof. On a small neighborhood in $\mathbb{R}^n$ take an orthonormal (adapted) moving frame $v_1, \ldots, v_n$ such that, at points in M, $v_1, \ldots, v_k$ are tangent to M. There is a unique matrix of 1-forms $\omega = (\omega_{ij})$ with $dv_j = \sum_{i=1}^n v_i\,\omega_{ij}$. Let $A = (v_1, \ldots, v_n) \in O(n)$ so, in matrix notation, $dA = A \cdot \omega$ or $\omega = A^{-1}\,dA$. Thus $(\omega_{ij})_{i,j=1,\ldots,k}$ is just the pull-back of the universal connection with respect to the moving frame. But this agrees with the connection induced from $\mathbb{R}^n$, since on M we have $\nabla v_j = \sum_{i=1}^k v_i\,\omega_{ij}$.

Theorem. *Let M be a smooth m-dimensional manifold with a k-dimensional vector bundle E. Suppose that E has a (fiber) metric h and a connection compatible with h. Then for $n \geqq \frac{1}{2}(k+m)^2 + 7(k+m) + 10)$ there exists a smooth map $f\colon M \to G_k(\mathbb{R}^n)$ such that E and $f^*\gamma^k(\mathbb{R}^n)$ are isomorphic as bundles with metrics and connections. If the metric and connection on E are real-analytic then the map f may be taken to be analytic.*

Proof. Let g be any Riemannian metric on M. (If M is analytic, we may choose g to be analytic.) Make the total space of E into a Riemannian manifold by taking the orthogonal direct sum of g and h; i.e. use g on horizontal vectors and h on vertical vectors. By the Nash imbedding theorem E imbeds isometrically in $\mathbb{R}^n$ where, according to Gromov-Rokhlin [1], we may take $n = \frac{1}{2}((k+m)^2 + 7(k+m) + 10)$. (This imbedding may be taken to be analytic if E is analytic.) Define a map $E \to G_k(\mathbb{R}^n)$ as follows: Given a point in E, take the tangent space (translated to the origin) to the fiber at its image in $\mathbb{R}^n$. Composing this with the zero section gives the desired map.

We must now show that the connection pulled back from the universal connection is the same as the connection that we started with. By the argument of the above proposition, the pull-back connection is the same as the one induced from $\mathbb{R}^n$, and hence the same as the one induced from the Riemannian structure of E. The proof is completed by the following proposition which shows that, as vector bundles with metrics and connections, E is canonically identified with the normal bundle to the zero-section.

Proposition. *Let $\tilde{\nabla}$ be the Riemannian connection on E. Then for $X \in TM \subset TE$,*

(1) *if Y is tangent to M, so is $\tilde{\nabla}_X Y$;*

(2) *if Y is normal to M, so is $\tilde{\nabla}_X Y$;*

(3) *the induced connections on the tangent and normal bundles to M agree with the original connections.*