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Tannaka-Krein duality for compact matrix pseudogroups. Twisted $SU(N)$ groups

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Summary. The notion of concrete monoidal W^*-category is introduced and investigated. A generalization of the Tannaka-Krein duality theorem is proved. It leads to new examples of compact matrix pseudogroups. Among them we have twisted $SU(N)$ groups denoted by $S_\mu U(N)$. It is shown that the representation theory for $S_\mu U(N)$ is similar to that of $SU(N)$: irreducible representations are labeled by Young diagrams and formulae for dimensions and multiplicity are the same as in the classical case.

0. Introduction

Two notions play the central role in this paper. The first one is the notion of compact matrix pseudogroup investigated in [8], the second introduced later in the next Section is the notion of concrete monoidal W^*-category. The two notions are linked in the following way. For any compact matrix pseudogroup G, the category of all unitary representations of G endowed with its natural structure is a concrete monoidal W^*-category. It satisfies certain conditions. All concrete monoidal W^*-categories satisfying these conditions can be obtained in this way. The last statement is a generalization of the Tannaka-Krein duality theorem. In special cases this result combined with Prop 2.4 and Theorem 1.5 of [8] gives the classical Tannaka-Krein theorem.

The generalized Tannaka-Krein duality gives us a large class of examples of compact matrix pseudogroups. The point is that there exists a procedure producing concrete monoidal W^*-categories which corresponds to the well known from the elementary algebra method of construction of semigroups (monoid) starting with given generators and relations. We shall describe this procedure in a special case leading to twisted $SU(N)$ groups.

The paper is organized in the following way. In Sect. 1 we remind the definition of compact matrix pseudogroup, list the properties of unitary representations and introduce the notion of concrete monoidal W^*-category. Then we

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formulate the duality theorem and give some applications. At this moment twisted $SU(N)$ groups are introduced. The proof of the duality theorem is presented in Sect. 3. Section 2 contains elementary facts concerning concrete monoidal W^*-categories that are used in the main proof. In Sect. 4 we investigate twisted $SU(N)$ groups and prove that their representation theory is fully similar to that of $SU(N)$.

Working with categories in a not completely trivial way one meets the logical problems related to the fact that there is no set containing all the objects of the category. In the context of the present paper these problems seem not to be serious. The objects related to any fixed Hilbert space form a set and we can restrict our considerations to Hilbert spaces belonging to some Hilbert Space Universe (i.e. to a set of Hilbert spaces containing all $\mathbb{C}^N$ and closed under direct sum, tensor product and passing to a subspace operations, cf. the Grothendieck notion of universal set [1]).

In the paper we use ① and Ⓣ products introduced in [7] and [8]. Let A be an algebra and K, L be $f-d.$ (finite dimensional) complex vector spaces. We remind that

$$\text{①}: (B(K)\otimes A)\times(B(K)\otimes A)\to B(K)\otimes A\otimes A$$
$$\text{Ⓣ}: (B(K)\otimes A)\times(B(L)\otimes A)\to B(K\otimes L)\otimes A$$

are bilinear mappings such that

$$(m_1\otimes a)\,\text{①}\,(m_2\otimes b)=m_1 m_2\otimes a\otimes b$$
$$(m\otimes a)\,\text{Ⓣ}\,(n\otimes b)=m\otimes n\otimes a b$$

for any $m_1, m_2, m\in B(K)$, $n\in B(L)$ and $a, b\in A$.

Assume that A is unital and that v and w belonging to $B(L)\otimes A$ and $B(K)\otimes A$ resp. are invertible. One can easily check that $v\,\text{Ⓣ}\,w$ is invertible and

$$(v\,\text{Ⓣ}\,w)^{-1}=(\tau\otimes\mathrm{id})(w^{-1}\,\text{Ⓣ}\,v^{-1}) \tag{0.1}$$

where $\tau\colon B(K\otimes L)\to B(L\otimes K)$ is the C^*-isomorphism such that $\tau(m\otimes n)=n\otimes m$ for any $m\in B(K)$ and $n\in B(L)$.

1. Definitions and main results

Let A be a C^*-algebra with unity. The set $M_N(A)$ of all $N\times N$ matrices with entries belonging to A will be identified with $B(\mathbb{C}^N)\otimes A$. Let $u=(u_{kl})_{kl=1,2,\ldots,N}$ be such a matrix. We remind (cf. [8]) that the pair $G=(A,u)$ is said to be a compact matrix pseudogroup if the following three conditions are satisfied.

CMP I. The *-subalgebra $\mathscr{A}$ generated by matrix elements of u is dense in A.

CMP II. There exists C^*-algebra homomorphism

$$\phi\colon A\to A\otimes A$$