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On the derived categories of coherent sheaves on some homogeneous spaces

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To A. Grothendieck on his 60-th birthday

$D^b(P^n)$, the derived category of coherent sheaves on the projective space P^n, was described in the papers of A.A. Beilinson [2] and I.N. Bernstein, I.M. Gelfand, S.I. Gelfand [5]. In the paper [2], two families of sheaves on P^n were distinguished: $\mathcal{O}(-i)$ and $\Omega^i(i)$, $i=0, 1, \ldots, n$. Sheaves from each of these families have no higher Ext's between each other and are "free generators" of the category $D^b(P^n)$. That means, that $D^b(P^n)$ is equivalent (as a triangulated category) to the homotopy category of finite complexes of sheaves, consisting of finite direct sums of $\mathcal{O}(-i)$ (resp. $\Omega^i(i)$), $i=0, 1, \ldots, n$.

In the present paper we describe, in a similar way, some more triangulated categories. More precisely, the description proceeds as follows. If $\mathfrak{A}$ is a preadditive category, then one can form a triangulated category $\mathrm{Tr}(\mathfrak{A})$, which is "generated freely" by $\mathfrak{A}$. Its objects are finite complexes, consisting of finite formal direct sums of objects of $\mathfrak{A}$, and morphisms are homotopy classes of morphisms of complexes. The description of a given triangulated category $\mathfrak{C}$ in the form $\mathrm{Tr}(\mathfrak{A})$ is practical enough, especially when the functor $\mathfrak{C} \to \mathrm{Tr}(\mathfrak{A})$ is given explicitly. We represent in the form $\mathrm{Tr}(\mathfrak{A})$ the derived categories of coherent sheaves on flag varieties and quadrics, and also the derived categories of finite-dimensional representations of parabolic subgroups in $GL(n, \mathbb{C})$.

In the §1 we fix the notations and recall the formulation of some facts from representation theory and homological algebra, which are necessary for the sequel. Of importance to us is the notion of a convolution, or of a total object of a finite complex over a triangulated category. This notion is needed to make some sense to the words "resolution in the derived category". Such a convolution is not canonical. Moreover, it even not always exists, the obstructions to its existence being the higher Massey compositions (or Toda brackets) of consecutive differentials in the complex. In the context of topological spaces or spectra such questions were treated in [23, 24].

In the §2 we give a general construction of "dual families" (such as $\{\Omega^i(i)\}_{i=0}^n$ for $\{\mathcal{O}(-i)\}_{i=0}^n$), and resolutions of the diagonal. There is also defined a triangulated category $\mathrm{Tr}(\mathfrak{A})$ for a differential graded (DG-) category $\mathfrak{A}$ and, under certain assumptions, a "duality theorem" is proved: $\mathrm{Tr}(\mathfrak{A})^{\mathrm{op}}$ is equivalent to

$\mathrm{Tr}(\hat{\mathfrak{A}})$, where $\hat{\mathfrak{A}}$ is the "cobar-category" for $\mathfrak{A}$. The consideration of DG-models for Ext-algebras (and categories) is a natural generalisation of the Priddy duality [18] between algebras with quadratic relations. This circumstance was indicated to the author by B.L. Feigin and V.V. Schechtman, to whom the author is sincerely grateful. The §2 was also influenced by recent papers of J.-M. Drezet [7] and A.L. Gorodentsev, A.N. Rudakov [8], in which, for each n, a series of pre-additive categories $\mathfrak{A}$ was constructed with the property $\mathrm{Tr}(\mathfrak{A}) \sim D^b(P^n)$.

In §3, the case of flag varieties is treated. In the particular case of Grassmann varieties (as well as for quadrics below, in §4) the corresponding pre-additive category will be a Koszul (in the sense of Priddy [18]) category with quadratic relations. It is not the case for general flag varieties.

In §4, we consider smooth projective quadrics. Our approach is based on the use of the graded Clifford algebra A, which is Priddy dual to the function algebra on the quadratic cone. Besides, we consider flag varieties $F(1, n-1, \mathbb{C}^n)$, which are incidence quadrics. We give for them another representation of the derived category of coherent sheaves in the form $\mathrm{Tr}(\mathfrak{A})$, where the category $\mathfrak{A}$ is Koszul. An approach very close to the ours was developed in the paper of R.G. Swan [19]. Author's result, announced in [12], was obtained independently.

Finally, in §5, the construction of §3 is applied to the category of finite-dimensional representations of parabolic subgroups in $GL(n, \mathbb{C})$. To do this, we consider the homogenous vector bundle on the flag variety, corrresponding to such a representation.

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§1. Preliminaries

1.1. For general facts about triangulated categories, see [10, 21]. The bounded derived category of an abelian category $\mathfrak{A}$ will be denoted $D^b(\mathfrak{A})$. The i times iterated translation functor in a triangulated category will be denoted $E \to E[i]$. On complexes (always supposed to be cohomological) this functor is defined by the formula $(C^{\cdot}[i])^j = C^{i+j}$. If E, F are objects of a triangulated category $\mathfrak{C}$, then $\mathrm{Ext}^i_{\mathfrak{C}}(E, F)$ means $\mathrm{Hom}_{\mathfrak{C}}(E, F[i])$. If $\mathfrak{A}$ is an additive category, then $\mathrm{Hot}(\mathfrak{A})$ is the homotopy category of bounded complexes over $\mathfrak{A}$. If X is a scheme, then the category of coherent sheaves on X is denoted $\mathrm{Sh}(X)$. The category $D^b(\mathrm{Sh}(X))$ is denoted simply $D^b(X)$. We denote identically algebraic vector bundles and locally free sheaves of their sections.

The category of covariant functors between categories $\mathfrak{C}$ and $\mathfrak{D}$ will be denoted $\mathrm{Fun}(\mathfrak{C}, \mathfrak{D})$, and of contravariant functors-$\mathrm{Fun}^o(\mathfrak{C}, \mathfrak{D})$.

1.2. Let $\mathfrak{C}$ be a triangulated category and $C^{\cdot}$ – a bounded complex over $\mathfrak{C}$, which we can suppose to be situated in degrees from 0 to n:

$$C^{\cdot} = \{C^0 \xrightarrow{d^0} \dots \xrightarrow{d^{n-1}} C^n\}.$$