The Radius of Convergence of Poincaré Series of Loop Spaces

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Abstract. Let $R_S$ (resp. $R_A$) be the radius of convergence of the Poincaré series of a loop space $\Omega(S)$ (resp. of the Betti-Poincaré series of a noetherian connected graded commutative algebra $A$ over a field $\mathbb{K}$ of characteristic zero).

If $S$ is a finite 1-connected CW-complex, the rational homotopy Lie algebra of $S$ is finite dimensional if and only if $R_S = 1$. Otherwise $R_S < 1$.

There is an easily computable upper bound (usually less than 1) for $R_S$ if $S$ is formal or coformal.

On the other hand $R_A = +\infty$ if and only if $A$ is a polynomial algebra and $R_A = 1$ if and only if $A$ is a complete intersection (Golod and Gulliksen conjecture). Otherwise $R_A < 1$ and the sequence $\dim \text{Tor}_p^H(\mathbb{K}, \mathbb{K})$ grows exponentially with $p$.

I. Introduction

I.1. Among the 1-connected topological spaces $S$, we distinguish two classes according to the dimension $\pi_*(S) \otimes \mathbb{Q}$. More precisely, $S$ is elliptic if $\dim \pi_*(S) \otimes \mathbb{Q} < +\infty$ and hyperbolic if $\dim \pi_*(S) \otimes \mathbb{Q} = +\infty$.

I.2. Let $R_S$ be the radius of convergence for Poincaré series of the loop space $\Omega(S)$. If $S$ elliptic, $R_S = 1$ or $+\infty$. As a consequence of [4] and [10] we have the

Theorem (II.6). If $H^*(S; \mathbb{Q})$ is a noetherian algebra, $S$ is hyperbolic if and only if $R_S < 1$.

It is moreover (II.8) easy to find a lower bound for $R_S$.

I.3. The next problem is to find, for a hyperbolic space $S$, an easily computable real number $r$, such that $R_S \leq r < 1$. The principal result of this paper is to find

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such an $r$ when $S$ is formal. (Recall that a space $S$ is formal if its rational-iziation $S_0$ has an automorphism $\psi$ satisfying $\psi^*(x)=x^{[1]} \cdot x$ for each $x$ belong- 

\[
\text{Theorem (III.4). Let } S \text{ be a space which is both hyperbolic and formal, and such that, } \dim H^*(S; \mathbb{Q}) < + \infty. \text{ Then } R_S \leq r \text{ where } r = \inf \left\{ |z_i|, \text{ } z_i \text{ running through the zeros of the Poincaré polynomial of } S, \sum_{i=0}^{+\infty} \dim H^i(S; \mathbb{Q}) t^i \right\}.
\]

There follows

\[
\text{Corollary (III.6). If } S \text{ is a } (p-1)\text{-connected compact formal oriented } n\text{-manifold, then } R_S \leq \left( \frac{n}{p b_p} \right)^{1/p} \text{ where } b_p \text{ is the } p\text{th Betti number of } S.
\]

I.4. The result of Theorem III.4 holds also under a weaker assumption namely for spaces with a homotopical weight decomposition (III.1). In a dual setting we study spaces equipped with a homological weight decomposition and ob-
tain an upper bound for $R_S$. Such spaces include those which are cofiber of a map between suspensions (formula VI.8).

I.5. In our second main result, we apply the same techniques in algebra and give a proof of the Golod and Gulliksen conjecture [19] for graded connected algebras and of conjecture $C_2$ Avramov [3]. More precisely, as a corollary of Theorem IV.5 we obtain:

\[
\text{Theorem (IV.7). Let } H \text{ be a noetherian and connected graded commutative algebra over } \mathbb{K}. \text{ If } R_H \text{ denotes the radius of convergence of the Betti-Poincaré series}
\]

\[
P_H(t) = \sum_{p \geq 0} \dim \operatorname{Tor}_p^H(\mathbb{K}, \mathbb{K}) t^p
\]

there are only three possibilities:

\begin{enumerate}
  \item $R_H = + \infty$, in this case $H$ is a polynomial algebra;
  \item $R_H = 1$, in this case $H$ is a complete intersection and \[
  \dim \operatorname{Tor}_p^H(\mathbb{K}, \mathbb{K}) \leq K p^m
  \]
  for some fixed $K$ and $m$;
  \item $R_H < 1$; in this case $H$ is not a complete intersection, and there exists a constant $C > 1$ such that \[
  \dim \operatorname{Tor}_p^H(\mathbb{K}, \mathbb{K}) \geq C^p
  \]
  for all $p$.
\end{enumerate}

I.6. The paper is organised as follows:

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