Convergence of Mixed Finite Element Approximations for the Shallow Arch Problem

B.D. Reddy
Department of Applied Mathematics, University of Cape Town, 7700 Rondebosch, South Africa

Summary. The stability and convergence of mixed finite element methods are investigated, for an equilibrium problem for thin shallow elastic arches. The problem in its standard form contains two terms, corresponding to the contributions from the shear and axial strains, with a small parameter. Lagrange multipliers are introduced, to formulate the problem in an alternative mixed form. Questions of existence and uniqueness of solutions to the standard and mixed problems are addressed. It is shown that finite element approximations of the mixed problem are stable and convergent. Reduced integration formulations are equivalent to a mixed formulation which in general is distinct from the formulation shown to be stable and convergent, except when the order of polynomial interpolation $t$ of the arch shape satisfies $1 \leq t \leq \min(2, r)$ where $r$ is the order of polynomial approximation of the unknown variables.

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1. Introduction

We investigate the efficacy of using finite element methods based on a mixed formulation, in the analysis of an equilibrium problem for thin shallow elastic arches. The Mindlin hypotheses are assumed and the problem in its standard form contains two terms, corresponding to the contributions from the shear and axial strains, with a small parameter. Lagrange multipliers corresponding to shear and axial force are introduced, to formulate the problem in an alternative mixed form. Questions of existence and uniqueness of solutions to the standard and mixed problems are addressed, after which finite element approximations of these problems are formulated. We are particularly interested here in the conditions under which the discrete mixed formulation is stable and convergent, and the conditions under which reduced integration formulations of the standard problem are equivalent to mixed formulations.

Reduced integration formulations have enjoyed considerable popularity in the engineering literature [4, 6–9], particularly in problems involving beams, plates and shells, and problems involving constraints such as incompressibility.
These formulations are often a means of overcoming the poor approximation (locking) experienced with full integration. In most cases there do not exist analyses justifying the use of reduced integration and providing guidelines for its use (two notable exceptions are problems with incompressibility [7] and the Timoshenko beam problem [1]), and practical rules have been set up on the basis of numerical experiments.

Less popular have been the use of mixed methods wherein the Lagrange multipliers are eliminated by using the fact that they are expressible as orthogonal projections of expressions involving the primitive variables. Such methods constitute a viable alternative to reduced integration methods if the orthogonal projections are onto spaces of functions which are discontinuous across element boundaries; the projections can then be calculated very rapidly since they involve the solution of a small set of linear equations.

We study a shallow arch problem which has both a shear and an axial term, so that there are two potential sources of locking. This investigation is in the spirit of that carried out by Arnold [1] on mixed methods for the Timoshenko beam, and by Kikuchi [5] on mixed methods for circular arches obeying the Kirchhoff assumptions. These two studies each involve one source of locking, the shear term in the beam problem and the axial term in the arch problem.

We show that finite element approximations of the mixed problem are stable and convergent. Reduced integration formulations are equivalent to a mixed formulation which in general is distinct from the formulation shown to be stable and convergent, except when the order of polynomial interpolation $t$ of the arch shape satisfies $1 \leq t \leq \min(2, r)$, where $r$ is the order of polynomial approximation of the unknown variables.

Under these circumstances, then, reduced integration techniques are stable and convergent. Nevertheless, the analogy between mixed and reduced integration methods would indicate that the latter ought to work relatively well in practice, for arbitrary interpolations of the arch shape.

We also discuss the viability of mixed methods in which the Lagrange multipliers are eliminated by projections at the element level.

2. Notation

We denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the inner product and associated norm on $(L^2(0, 1)$. The Sobolev space of functions which together with their weak derivatives of order $\leq m$ are square-integrable on $(0, 1)$, is denoted by $H^m(0, 1)$; the inner product $\langle \cdot, \cdot \rangle_m$ and norm $\| \cdot \|_m$ on $H^m(0, 1)$ are defined by

$$
\langle u, v \rangle_m = \sum_{k=0}^{m} \frac{d^k u}{ds^k} \frac{d^k v}{ds^k},
$$

$$
\| u \|_m = (u, u)^{\frac{1}{2}},
$$

for $u, v \in H^m(0, 1)$. We will also require the space $H^1_0(0, 1)$, defined by $H^1_0(0, 1) = \{ v \in H^1(0, 1) : v(0) = v(1) = 0 \}$. The Poincaré inequality asserts that

$$
\| u \| \leq \| u' \|, \quad u \in H^1_0(0, 1),
$$