Liapunov Functions and Error Bounds for Approximate Solutions of Ordinary Differential Equations

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Summary. It is shown that Liapunov functions may be used to obtain error bounds for approximate solutions of systems of ordinary differential equations. These error bounds may reflect the behaviour of the error more accurately than other bounds.

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1. Introduction

For $t_0 < t_1 < \cdots < t_N$ let $y_0, y_1, \ldots, y_N$ be approximations to $x(t_0), x(t_1), \ldots, x(t_N)$, where $x$ is the solution of the system of $n$ real differential equations

$$x' = f(x), \quad x(t_0) = x_0.$$ 

Define an approximation $y$ which is piecewise continuously differentiable on $[t_0, t_N]$ and is such that $y(t_i) = y_i$, $i = 0, 1, \ldots, N$. The error $u = x - y$ is the continuous solution of

$$u' = f(y + u) - y', \quad u(t_0) = x_0 - y_0. \quad (1.1)$$

Dahlquist [2] showed that an 'a posteriori' bound for the error may be obtained as the solution of a single linear differential equation and Cooper [1] modified this to obtain an error bound as the solution of a Riccati equation. In each case the differential equation for the error bound depends on a logarithmic norm which may be positive even when the system of differential equations (1.1) is asymptotically stable. In this case, in particular, transformations may improve the bounds but it is not clear how such transformations should be chosen. This article shows that time independent Liapunov functions may be used to obtain error bounds and this shows how the transformations should be chosen. This approach is more flexible and can give appreciably better bounds.

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Let $D \subseteq \mathbb{R}^n$ be a convex domain with $y \in D$ and $x \in D$ for all $t \in [t_0, t_N]$. Suppose that the Jacobian matrix of $f$ satisfies the Lipschitz condition

$$\|J(w) - J(z)\| \leq 2k\|w - z\| \quad \forall w, z \in D,$$

for the vector norm $\|w\| = \sqrt{w^H w}$ and the corresponding spectral matrix norm $\|A\| = \sqrt{\lambda[A^H A]}$, where $\lambda[H]$ denotes the maximum eigenvalue of $H$. The error $u$ is the solution of

$$u' = J(y)u + r(t; u) + d(t), \quad u(t_0) = x_0 - y_0,$$

where $r(t; u) = f(y + u) - f(y) - J(y)u$ and where $d(t) = f(y) - y'$. It may be shown [1, p. 164] that $\|r(t; u)\| \leq k\|u\|^2$. Let $H(t) = \frac{1}{2}[J(y)^H + J(y)]$. The main result obtained by Cooper [1] shows that a bound for the error is the solution of a Riccati equation. Given $e_0 \geq \|u(t_0)\|$, there is a $t^* \in (t_0, t_1]$ such that $e \geq 0$ for $t \in [t_0, t^*)$ where

$$e' = ke^2 + \lambda[H(t)]e + \|d(t)\|, \quad e(t) = e_0.$$

Here $\lambda[H(t)]$ is the logarithmic norm of $J(y)$ and may be positive even though the reduced linear equation $u' = J(t)u$ is asymptotically stable. In practice the coefficients of the Riccati equation have to be bounded on sub-intervals and here it is convenient for comparison purposes to choose the intervals $(t_i, t_{i+1})$, $i = 0, 1, \ldots, N - 1$. Let

$$\eta_i = \sup_{t \in (t_i, t_{i+1})} \|y(t) - y_i\|, \quad d_i = \sup_{t \in (t_i, t_{i+1})} \|d(t)\|, \quad i = 0, 1, \ldots, N - 1.$$

It may be shown that if $e_0 \geq \|u(t_0)\|$ then $e \geq 0$ on some $[t_0, t^*)$ where $e$ is the continuous solution of the sequence of Riccati equations

$$e' = ke^2 + (\lambda[H(t_i)] + 2k\eta_i)e + d_i, \quad i = 0, 1, 2, \ldots, \quad e(t_0) = e_0. \quad (1.2)$$

In this article it is shown that a similar sequence of equations may be obtained by using Liapunov functions and that equations (1.2) are a special case.

2. Error Bounds Using Liapunov Functions

Suppose that an error bound $e \geq \|u\|$ has been obtained on $[t_0, t_i]$. For $t > t_i$ the error is the solution of

$$u' = Au + [J(y) - A]u + r(t; u) + d(t), \quad u(t_i) = x(t_i) - y_i, \quad (2.1)$$

where, for example, $A = J(y_i)$ may be chosen. Suppose that the linear equation $u' = Au$ is asymptotically stable. Then, Hahn [3, p. 115], for any negative definite Hermitian matrix $C$, there is a positive definite Liapunov function $v = u^H Bu$ such that $v' = 2u^H Cu$ and $A^H B + BA = 2C$. 