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Summary. In this paper we study the stability and convergence properties of Bergman kernel methods, for the numerical conformal mapping of simply and doubly-connected domains. In particular, by using certain well-known results of Carleman, we establish a characterization of the level of instability in the methods, in terms of the geometry of the domain under consideration. We also explain how certain known convergence results can provide some theoretical justification of the observed improvement in accuracy which is achieved by the methods, when the basis set used contains functions that reflect the main singular behaviour of the conformal map.

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1. Introduction

Let \( \partial \Omega \) be a closed piecewise analytic Jordan curve in the complex \( z \)-plane, assume that 0 is in \( \Omega = \text{Int}(\partial \Omega) \), and let \( f \) be the function which maps conformally \( \Omega \) onto the unit disc \( \{ w : |w| < 1 \} \) so that \( f(0)=0 \) and \( f'(0)>0 \). Also, let \( L^2(\Omega) \) be the Hilbert space of all square integrable analytic functions in \( \Omega \), denote by \( \langle \cdot, \cdot \rangle \) the inner product of \( L^2(\Omega) \), i.e.

\[
\langle u, v \rangle = \iint_{\Omega} u(z) \overline{v(z)} \, dS_z,
\]

and let \( K(\cdot, 0) \) be the Bergman kernel function of \( \Omega \). Then, the kernel \( K(\cdot, 0) \) is uniquely characterized by the reproducing property

\[
\langle g, K(\cdot, 0) \rangle = g(0), \quad \forall g \in L^2(\Omega),
\]
and is related to the mapping function $f$ by means of

$$f'(z) = \left\{ \frac{\pi}{K(0,0)} \right\}^{\frac{1}{2}} K(z,0); \quad (1.3)$$

see e.g. [1, 7, 8, 12].

Let $\eta_j$, $j=1,2,3,\ldots$, be a complete set of functions of $L_2(f_\Omega)$. Then the reproducing property (1.2) and the relation (1.3) suggest the following procedure for approximating the mapping function $f$. The set $\{\eta_j\}_{j=1}^n$ is orthonormalized by means of the Gram-Schmidt process to give the orthonormal set $\{\eta_j^*\}_{j=1}^n$. The kernel $K(z,0)$ is then approximated by the finite Fourier series sum

$$K_n(z,0) = \sum_{j=1}^n \langle K(\cdot,0), \eta_j^* \rangle \eta_j^*(z)$$

and finally equation (1.3) is used to give the approximation

$$f_n(z) = \left\{ \frac{\pi}{K_n(0,0)} \right\}^{\frac{1}{2}} \int_0^z K_n(\zeta,0) d\zeta,$$

(1.5)

to the function $f$. In other words the approximation $f_n$ is obtained after first determining the least squares approximation, in

$$A_n = \text{span} \{\eta_1, \eta_2, \ldots, \eta_n\},$$

(1.6)
to the Bergman kernel function $K(\cdot,0)$. This method of approximating $f$ is the well-known Bergman kernel method (BKM); see e.g. [1, 2, 4, 7, 8, 10, 12–14].

Let now $\partial \Omega_1$ and $\partial \Omega_2$ be two closed piecewise analytic Jordan curves such that $\partial \Omega_1 \subset \text{Int}(\partial \Omega_2)$ and $0 \in \text{Int}(\partial \Omega_1)$, denote by $\Omega$ the doubly-connected domain

$$\Omega = \text{Ext}(\partial \Omega_1) \cap \text{Int}(\partial \Omega_2),$$

(1.7)

and let $f$ be the function which maps conformally $\Omega$ onto a circular annulus $\{w: 1 < |w| < M\}$ so that $f(\zeta_1) = 1$, where $\zeta_1$ is some fixed point on $\partial \Omega_1$. Also let

$$H(z) = f'(z)/f(z) - 1/z,$$

(1.8)

and denote by $L^2_2(\Omega)$ the Hilbert space of all functions in $L_2(\Omega)$ which also possess a single-valued indefinite integral in $\Omega$. Then, it can be shown that for $\eta \in L^2_2(\Omega)$

$$\langle \eta, H \rangle = i \int_{\partial \Omega_1 \cup \partial \Omega_2} \eta(z) \log |z| dz,$$

(1.9)

provided that the function $\eta$ satisfies certain boundary continuity requirements; see [7, p.249] and the remark in [15, §2, p.686]. In other words, the determination of $\langle \eta, H \rangle$ does not require the explicit knowledge of $H$ and, because of this, an approximation $f_n$ to $f$ can be determined by means of (1.8), in a manner similar to the BKM. That is, the approximation $f_n$ to the conformal map of the doubly-connected domain (1.7) is determined from the