Inventiones math. 49, 121 – 135 (1978)

Inventiones mathematicae

On the Holonomic Systems of Linear Differential Equations, II *

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In this paper we shall study the restriction of holonomic systems of differential equations.

Let X be a complex manifold and Y a submanifold, and let $\mathcal{O}_X$ and $\mathcal{D}_X$ be the sheaf of the holomorphic functions and the sheaf of the differential operators of finite order, respectively. If a function u on X satisfies a system of differential equations, the restriction of u onto Y also satisfies the system of differential equations derived from the system on X. This leads to the following definition. Let $\mathcal{M}$ be a $\mathcal{D}_X$-Module. The restriction of $\mathcal{M}$ onto Y is, by definition, $\mathcal{O}_Y \underset{\mathcal{O}_X}{\otimes} \mathcal{M}$. In [4] it is proved that if $\mathcal{M}$ is a coherent $\mathcal{D}_X$-Module and if Y is non-characteristic to $\mathcal{M}$, then the restriction of $\mathcal{M}$ is also a coherent $\mathcal{D}_Y$-Module. However, if Y is characteristic, the restriction is no longer coherent in general. For examples, if $X=\mathbb{C}^n$ and $Y=\{x=(x_1, \ldots, x_n)\in X;\ x_1=0\}$ and $\mathcal{M}=\mathcal{D}_X$, the restriction $\mathcal{M}/x_1\mathcal{M}$ is a free $\mathcal{D}_Y$-Module generated by $D_1^m (m=0,1,2,\ldots)$ and is not coherent.

We shall prove the following theorems in this paper.

Theorem. *Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-Module on a complex manifold X and f a holomorphic map from Y to X. Then $\mathcal{O}_Y \underset{f^{-1}\mathcal{O}_X}{\otimes} f^{-1}\mathcal{M}$ is a holonomic system on Y.*

This theorem is proved by Bernstein [1] in the polynomial case.

At the same time, we shall prove

Theorem. *If $\mathcal{M}$ is a holonomic $\mathcal{D}_X$-Module, and if $\mathcal{I}$ is a coherent Ideal of $\mathcal{O}_X$, then $\varinjlim_m \mathcal{E}xt^k_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^m; \mathcal{M})$ are also holonomic $\mathcal{D}_X$-Modules.*

Theorem. *If $\mathcal{M}$ is a holonomic $\mathcal{D}_X$-Module defined on X and holonomic outside an analytic subset Y, then $\mathcal{M}/\mathcal{H}^0_Y(\mathcal{M})$ is holonomic on X.*

These theorems imply in particular the following: Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-Module and let ∇ be a meromorphic integrable connection on $\mathcal{F}$ with a pole

* This is the second of the series of papers which are concerned with holonomic systems. The paper [5] is the first of this series

0020/9910/78/0049/0121/$03.00

on a hypersurface Y. Then, $\mathscr{H}^0_{[X|Y]}(\mathscr{F})$ (i.e., the sheaf of the meromorphic sections of $\mathscr{F}$ with a pole on Y) is a holonomic $\mathscr{D}_X$-Module (in particular, coherent).

Also, we shall prove the following theorem.

Theorem. *For two holonomic $\mathscr{D}_X$-Modules $\mathscr{M}$ and $\mathscr{N}$, $\mathscr{E}xt^j(\mathscr{M};\mathscr{N})$ are constructible (i.e., $\dim_{\mathbb{C}}\mathscr{E}xt^j(\mathscr{M};\mathscr{N})_x<\infty$ for any $x\in X$ and there is a stratification on X on each of whose stratum $\mathscr{E}xt^j_{\mathscr{D}}(\mathscr{M},\mathscr{N})$ is locally constant).*

However, the author does not know how to stratify X so that $\mathscr{E}xt^j_{\mathscr{D}}(\mathscr{M},\mathscr{N})$ is constructible on the strata. This problem is tightly connected with the problem of determining the characteristic variety of $\mathscr{O}_Y\underset{\mathscr{O}_X}{\otimes}\mathscr{M}$.

I wish to thank J.E. Björk, J.-M. Kantor and B. Malgrange for their kind suggestions and friendly discussions about these subjects.

§ 1. Algebraic Local Cohomologies

1.1. In this paper we denote by X a complex manifold, by $\mathscr{O}_X$ the sheaf of the holomorphic functions on X and by $\mathscr{D}_X$ the sheaf of the linear differential operators of *finite order*.

1.2. Let $\mathscr{I}$ be a coherent $\mathscr{O}_X$-Ideal and Y the support of $\mathscr{O}_X/\mathscr{I}$. For an $\mathscr{O}_X$-Module $\mathscr{F}$, we define with [2, 3]

$$(1.2.1)\quad \Gamma_{[X|Y]}(\mathscr{F})=\varinjlim_m \mathscr{H}om_{\mathscr{O}_X}(\mathscr{I}^m;\mathscr{F}),$$

$$(1.2.2)\quad \Gamma_{[Y]}(\mathscr{F})=\varinjlim_m \mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_X/\mathscr{I}^m;\mathscr{F}).$$

This definition depends only on Y (not on the choice of $\mathscr{I}$). We have an exact sequence:

$$(1.2.3)\quad 0\to\Gamma_{[Y]}(\mathscr{F})\to\mathscr{F}\to\Gamma_{[X|Y]}(\mathscr{F}).$$

Lemma 1.1. *If $\mathscr{F}$ is a $\mathscr{D}_X$-Module, $\Gamma_{[X|Y]}(\mathscr{F})$ and $\Gamma_{[Y]}(\mathscr{F})$ have a structure of $\mathscr{D}_X$-Modules so that* (1.2.3) *is* $\mathscr{D}_X$-linear.

Proof. We have evidently

$$\Gamma_{[X|Y]}(\mathscr{F})=\varinjlim_m \mathscr{H}om_{\mathscr{D}_X}(\mathscr{D}_X\mathscr{I}^m;\mathscr{F})$$

and

$$\Gamma_{[Y]}(\mathscr{F})=\varinjlim_m \mathscr{H}om_{\mathscr{D}_X}(\mathscr{D}_X/\mathscr{D}_X\mathscr{I}^m;\mathscr{F})$$

because $\mathscr{D}_X$ is faithfully flat over $\mathscr{O}_X$.

We shall define the multiplication of a differential operator P with $\Gamma_{[X|Y]}(\mathscr{F})$. Suppose that P is of order $\leqq l$. Then we have

$$\mathscr{D}_X\mathscr{I}^m P\subset\mathscr{D}_X\mathscr{I}^{m-l}\qquad\text{for } m\geqq l.$$