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Homology Fibrations and the "Group-Completion" Theorem

D. McDuff (York) and G. Segal (Oxford)

A topological monoid M has a classifying-space BM, which is a space with a base-point. There is a canonical map of H-spaces $M \to \Omega BM$ from M to the space of loops on BM, and it is a homotopy-equivalence if the monoid of connected components $\pi_0 M$ is a group. The "group-completion" theorem ([2–4, 6, 9]) describes the relationship between M and ΩBM in general. Let us regard $\pi = \pi_0 M$ as a multiplicative subset of the Pontrjagin ring $H_*(M)$, using singular integral homology. The map $M \to \Omega BM$ induces a homomorphism of Pontrjagin rings, and (because $\pi_0(\Omega BM)$ is a group) the image of π in $H_*(\Omega BM)$ consists of units.

Proposition 1. *If π is in the centre of $H_*(M)$ then*

$$H_*(M)[\pi^{-1}] \xrightarrow{\cong} H_*(\Omega BM).$$

Although several proofs of this theorem have appeared its importance for the process of "Quillenization"[1] perhaps justifies our publishing the present one, which is simple and conceptual. We shall prove, moreover, a stronger statement than Proposition 1 in the two respects described in Remarks 1 and 2 below. Our method was suggested by Quillen's second unpublished proof, and by conversations with him for which we are very grateful. The use of homology fibrations arose from [5]. We have listed some examples and applications of the theorem at the end.

Remark 1. In Proposition 1 one need not assume that π is in the centre of $H_*(M)$, but only that $H_*(M)[\pi^{-1}]$ can be constructed by right fractions. Recall that if π is a multiplicative subset of a ring A one says that $A[\pi^{-1}]$ can be constructed by right fractions if every element of it can be written ap^{-1} with $a \in A$, $p \in \pi$, and if $a_1 p_1^{-1} = a_2 p_2^{-1}$ if and only if $a_1 p'_1 = a_2 p'_2$ and $p_1 p'_1 = p_2 p'_2$ for some $p'_1, p'_2 \in \pi$. A typical example is when π consists of the powers of an element $x \in A$ such that $ax = x\alpha(a)$ for all $a \in A$, where α is an endomorphism of A. This arises as the Pontrjagin ring of the monoid of all maps $S^n \to S^n$ whose degrees are powers of a prime p, as we shall see below.

[1] This word is due to I.M. Gel'fand.

We shall prove Proposition 1 by constructing a space M_∞ whose homology is obviously $H_*(M)[\pi^{-1}]$, and a homology equivalence $M_\infty \to \Omega BM$. The basic example is the case when $M = \coprod_{n \geq 0} B\Sigma_n$, where Σ_n is the n^{th} symmetric group, and the monoid structure of M comes from juxtaposition $\Sigma_n \times \Sigma_m \to \Sigma_{n+m}$. Then M_∞ will be $\mathbb{Z} \times B\Sigma_\infty$.

Remark 2. To say that a map $f\colon X \to Y$ is a homology equivalence may have at least two meanings. The weaker one is that f induces an isomorphism of integral homology. The stronger is that $f_*\colon H_*(X; f^*A) \xrightarrow{\cong} H_*(Y; A)$ for every coefficient system A of abelian groups on Y. The map $M_\infty \to \Omega BM$ we shall construct will be a homology equivalence in the stronger sense. Thus ΩBM, whose components have of course abelian fundamental groups, is a "Quillenization" of M_∞. The advantage of allowing twisted coefficient systems is that one can conclude that $\tilde{M}_\infty \to \widetilde{\Omega BM}$ is a homology equivalence as well as $M_\infty \to \Omega BM$, where $\widetilde{\Omega BM}$ is the universal covering space of ΩBM, and $\tilde{M}_\infty$ is its pull-back to M_∞. This means that the fundamental group of $\tilde{M}_\infty$ must be perfect, and so our method incorporates a general proof that the commutator subgroup of $\pi_1(M_\infty)$ is perfect. If isolated this would reduce to Wagoner's argument in [11].

Everything we say below is true if homology equivalence is given either of the above meanings. Nevertheless it will be convenient to adopt a middle definition, allowing only *abelian* coefficient systems A on Y, i.e. those such that for each $y \in Y$ the group of automorphisms of the coefficient group A_y at y induced by the action of $\pi_1(Y, y)$ is abelian. Of course any system coming from ΩBM is abelian.

Our main idea is that of a *homology fibration.* In [5] a homology fibration was defined as a map $p\colon E \to B$ such that for each $b \in B$ the natural map $p^{-1}(b) \to F(p, b)$ from the fibre at b to the homotopical fibre at b is a homology equivalence. ($F(p, b)$ is defined as the fibre-product $P_b \times_B E$, where P_b is the space of paths in B beginning at b.) In this language to obtain a homology equivalence $M_\infty \to \Omega BM$ it is enough to produce a homology fibration $E \to BM$ with E contractible and with fibre M_∞ at the base-point.

If M is a topological group which acts on a space X one often considers the space X_M fibred over BM with fibre X, associated to the universal bundle $EM \to BM$. But the construction of X_M makes sense even if M is only a topological monoid, for X_M can be described as the realization of the topological category whose space of objects is X and whose space of morphisms is $M \times X$, a pair (m, x) being thought of as a morphism from x to mx. (Here, and in constructing BM also, we use the "thick" realization of simplicial spaces, denoted by $\| \quad \|$ in the appendix to [9].)

Our main result is

Proposition 2. *If M is a topological monoid which acts on a space X, and for each $m \in M$ the map $x \mapsto mx$ from X to itself is a homology equivalence, then $X_M \to BM$ is a homology fibration with fibre X.*

This should be compared with the fact that if $x \mapsto xm$ is a homotopy equivalence for each m then $X_M \to BM$ is a quasifibration. (When M is discrete this is a particular case of [7] (Lemma p. 98); in general it is a particular case of [9] (1.5).)