A Comparison of Some Bounds 
for the Nontrivial Eigenvalues of Stochastic Matrices

Christoph Zenger

Received October 20, 1971

Summary. Some recently published bounds for the nontrivial eigenvalues of stochastic matrices [1, 2, 4, 5] are compared. It is shown that the Deutsch bound [1] is the best of these bounds and is only slightly improved by a bound given in [3].

In [5] Schaefer gave a bound for the nontrivial eigenvalues of stochastic matrices:

Let $m$ be the smallest and $M$ be the largest element of the row-stochastic matrix $A$ of order $n$ with row-sum $s$.

Then the nontrivial eigenvalues $\lambda$ of $A$ satisfy

$$|\lambda| \leq \min (s - n \cdot m, n \cdot M - s).$$

Brauer [2] improved this bound:

Let $m_v$ be the minimum and $M_v$ be the maximum of the elements of the $v$-th column ($v = 1, 2, \ldots, n$) and

$$t = \sum_{v=1}^{n} m_v, \quad T = \sum_{v=1}^{n} M_v$$

then

$$|\lambda| \leq \min (s - t, T - s).$$

We show that both bounds are weaker than the Deutsch bound [1]

$$|\lambda| \leq b(A) = \frac{1}{2} \max_{i,j} \sum_{k} |a_{ik} - a_{jk}|.$$ 

Indeed we have $\frac{1}{2} |u - v| = \max (u, v) - \frac{1}{2} (u + v)$, therefore

$$b(A) = \max_{i,j} \sum_{k} \left( \max (a_{ik}, a_{jk}) - \frac{1}{2} (a_{ik} + a_{jk}) \right)$$

$$= \max_{i,j} \sum_{k} \max (a_{ik}, a_{jk}) - s$$

$$\leq \sum_{k} \max_{i} a_{ik} - s = T - s.$$ 

1 These are the eigenvalues not belonging to the eigenvector $(1, 1, \ldots, 1)^T$. 
From \( \frac{1}{2} |u - v| = \frac{1}{2} (u + v) - \min (u, v) \),
\[
b(A) = \max_{i, j} \left( \frac{1}{2} (a_{ik} + a_{jk}) - \min (a_{ik}, a_{jk}) \right)
\]
we get the other inequality in the same manner.

The proof shows that it is sufficient to take the maximum value \( M_r \) in Brauer's bound from only two rows instead of all rows, where the two rows are chosen such that the corresponding \( T \) is maximal. Clearly the analogous consideration holds for the minimum values \( m_r \). If \( \tilde{T} \) and \( \tilde{t} \) are determined in this way then the above proof shows that
\[
s - \tilde{t} = \tilde{T} - s = \min (s - \tilde{t}, \tilde{T} - s) = b(A)
\]
and the modified Brauer bound coincides with the Deutsch bound.

Recently Hadeler gave some bounds for the nontrivial eigenvalues of positive operators \([4]\). Applying his results to the case of stochastic matrices he got the following bounds:
\[
|h| \leq h_1(A) := s - n \cdot m
\]
\[
|h| \leq h_2(A) := \min \max_{i} \sum_{k} |a_{ik}a_{i1} - a_{ij}a_{jk}|
\]
\[
|h| \leq h_3(A) := \min \max_{i} \sum_{k} |a_{ik} - a_{jk}|.
\]

\( h_1(A) \) is strengthened by the Schaefer bound. Moreover we have \( h_2(A) \geq h_3(A) \geq b(A) \) for all stochastic matrices.

The second inequality was already stated by Hadeler. The first inequality is a consequence of the relations
\[
h_2(A) \geq \min \max_{i} \sum_{k} \sum_{l} |a_{ik}a_{il} - a_{fk}a_{il}|
\]
\[
= \min \max_{i} \sum_{k} |a_{ik} - a_{jk}| = h_3(A).
\]

A more detailed investigation of the situation of the nontrivial eigenvalues of stochastic matrices is given in \([3]\). In this paper a set \( G(A) \) of complex numbers (actually a generalized Bauer field of values) containing all nontrivial eigenvalues of \( A \) is constructed. For the corresponding numerical radius \( r(A) = \max_{z \in G(A)} |z| \) we have
\[
r(A) = \frac{1}{2} \max_{i, j} \left( a_{ii} + a_{jj} - a_{ij} - a_{ji} + \sum_{k \neq i, j} |a_{ik} - a_{jk}| \right).
\]
Obviously we have \( r(A) \leq b(A) \) for all row-stochastic \( A \). Moreover in \([3]\) an example for a matrix \( A \) is given with \( r(A) < b(A) \).

Thus \( r(A) \) is the best of all bounds considered in this paper.

References