A0-stability and Stiff Stability of Brown's Multistep Multiderivative Methods

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Summary. Brown [1] introduced k-step methods using l derivatives. Necessary and sufficient conditions for A0-stability and stiff stability of these methods are given. These conditions are used to investigate for which k and l the methods are A0-stable. It is seen that for all k and l with k < 1.5 (l + 1) the methods are A0-stable and stiffly stable. This result is conservative and can be improved for l sufficiently large. For small k and l A0-stability has been determined numerically by implementing the necessary and sufficient condition.

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1. Introduction

We shall consider numerical methods for solving the initial value problem

\[ y'(x) = f(x, y(x)), \quad y(a) = \eta \]  

which use the higher derivatives \( f^{(j)}(x, y) \), which are defined by \( f^{(0)}(x, y) = f(x, y) \) and

\[ f^{(j)}(x, y) = \frac{\partial}{\partial x} f^{(j-1)}(x, y) + \frac{\partial}{\partial y} f^{(j-1)}(x, y) f(x, y), \quad j = 1, 2, \ldots. \]

Let \( h > 0 \) be the stepsize, \( x_m = a + mh \), and \( y_m \) be approximations to the exact solution \( y(x_m) \) of (1). Moreover, let \( f_{m}^{(j)} = f^{(j)}(x_m, y_m) \). The methods are of the form

\[ \sum_{i=0}^{k} \alpha_i y_{n+i} = \sum_{j=1}^{l} h^j \beta_j f_{n+k}^{(j-1)} \quad \text{with} \quad \sum_{i=0}^{k} |\alpha_i| > 0, \quad \sum_{j=1}^{l} |\beta_j| > 0, \]  

\( n=0,1,2,\ldots \). Here \( \alpha_i \) and \( \beta_j \) are constants. For convenience in notation we
introduce the natural convention that $\beta_0 = -\alpha_k$. We shall always assume that the coefficients $\alpha_j, \beta_j$ satisfy

$$(-1)^{j+1} \beta_j \geq 0, \quad j = 0, 1, \ldots, l$$

$$\alpha_k \neq 0, \quad \beta_1 \neq 0$$

$$\alpha_0 + \alpha_1 + \ldots + \alpha_k = 0.$$  

(3a) \hspace{1cm} (3b) \hspace{1cm} (3c)

A method of form (2) is said to have error order $p$ if

$$\sum_{i=0}^{k} \alpha_i y(x + i h) - \sum_{j=1}^{l} h^j \beta_j y^{(j)}(x + k h) = C_{p+1} h^{p+1} y^{(p+1)}(x) + O(h^{p+2}),$$

$$C_{p+1} \neq 0,$$

for all sufficiently often differentiable functions $y(x)$. An important subclass of methods of form (2) satisfying (3) is obtained if one selects $\alpha_i$ and $\beta_j$ such that the method has highest possible error order. These methods have been introduced by Brown [1] and in Jeltsch, Kratz [15] it was shown that

$$\alpha_i = (-1)^{k-i} \binom{k}{i} (k-i)^{-l}, \quad i = 0, 1, \ldots, k-1$$

(4a)

$$\beta_j = (-1)^j / j! \sum_{i=0}^{k-1} (-1)^{k-i} \binom{k}{i} (k-i)^{-l}, \quad j = 0, 1, \ldots, l.$$  

(4b)

In [15] it was shown that the methods given by (2) and (4) have the error order $p = k + l - 1$ and that there is no method of form (2) with a higher error order. We shall call the formula (2) with (4) Brown's $(k, l)$-method. It was discussed in [15] for which $k$ and $l$ Brown's $(k, l)$-method is stable and for which it is not. Since the method belongs to the class of $k$-step methods with $l$ derivatives and $p \geq 1$ it converges if and only if it is stable (see e.g. Griepentrog [8], Spijker [17]). Convergence is also shown for strongly stable methods in Brown [1, 2]. Assuming strong stability an estimate for the global discretization error as well as the first term in the asymptotic error expansion are given in [1, 2]. In the present article we give necessary and sufficient conditions for the methods to be $A_0$-stable. These conditions are then used to investigate for which $k$ and $l$ Brown's $(k, l)$-methods are $A_0$-stable, see Fig. 1. For $k$ and $l$ small this has been done by implementing the criterias on the computer. It is proved that Brown's $(k, l)$-methods are $A_0$-stable and stiffly stable if $k \leq 1.5 (l+1)$. For any $\alpha < \alpha_j \sim 3.59112148$ it is $A_0$-stable and stiffly stable for $k \leq \alpha (l+1)$ and $l$ sufficiently large. Computer results suggest that $l \geq 2$ is already large enough. Moreover, these numerical results support the conjecture, that a zero-stable Brown's method is $A_0$-stable and stiffly stable. Finally we would like to mention that Brown's $(k, l)$-methods are the well known Backward Differentiation Methods BDM. In Liniger [16] it has been shown that the BDM are $A_0$-stable and strongly stable for $k \leq 5$. 
