A Constructive Method of Solving the Liapounov Equation for Complex Matrices*

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Summary. This paper describes a method of solving the Liapounov equation (1) \[ HM + M^*H = 2D, \]
where \( M \) is in upper Hessenberg form, \( D \) diagonal. Initialising the first row of the matrix \( A \) arbitrarily, one can find (by solving equations with one unknown) the unknown elements of \( A \) such that (2) \[ A M + M^*A^* = 2F, \]
where \( A \) differs from a Hermitian matrix only in that its diagonal elements need not be real. \( F \) is a diagonal matrix which is uniquely determined by the first row of \( A \). By solving Eq. (2) for several initial values one may generate several matrices \( A \) and \( F \) (in the most unfavourable case \( 2n-1 \) \( A \)'s and \( F \)'s are needed) and superpose them to get \( n \) linearly independent Hermitian matrices \( H_i \) and \( D_i \) respectively for which \[ H_iM + M^*H_i = 2D_i \]
is valid. Then one can solve the real system \[ \sum_{i=1}^{n} \lambda_j H_j = D \]
to obtain the solution \[ H = \sum_{j=1}^{n} \lambda_j H_j \]
of Eq. (1).

1. Introduction

In [1] it is shown how the stability problem can be solved for real \( n \times n \) matrices. This involves a method of solving the Liapounov equation \( SM + M^*S = I \) for real \( n \times n \) Hessenberg matrices \( M \). Programming this method yielded very satisfactory results, and so it was felt desirable to extend it to complex matrices.

2. Preliminaries

Let \( \mathbb{R} \) be the field of real numbers, \( \mathbb{C} \) the field of complex numbers. Let \( M \in L(\mathbb{C}^n) \), i.e. a complex \( n \times n \) matrix, and let \( \lambda_1, \ldots, \lambda_n \) be the eigenvalues of \( M \). The following theorem is then valid ([2], §4, p. 78):

**Theorem 1.** If \( \Delta(M) := \prod_{i,j=1}^{n} (\lambda_i + \bar{\lambda}_j) \neq 0 \) and \( P \) is a given Hermitian positive-definite matrix, then there exists a unique \( H \) satisfying \( HM + M^*H = P \), and \( H \) is Hermitian.

**Remark.** \( HM + M^*H = 0 \) has only the trivial solution \( H = 0 \) iff \( \Delta(M) \neq 0 \) is valid ([2], §2, p. 75).

For the sake of simplicity we will specially choose \( P = 2D \), \( D \) diagonal, in the following. \( D \) is then real because \( D^* = D \). The following definitions and lemmas are direct generalizations of the results for real matrices in [1], §2.

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Definition. A Hermitian matrix $H$ and a real diagonal matrix $D$ are called a Liapounov pair with respect to $M$ if they satisfy the equation

$$HM + M^* H = 2D. \quad (1)$$

Lemma 1. Two matrices $H$ and $D$, $H = H^*$, $D = D^*$ diagonal, form a Liapounov pair with respect to $M$ iff there is a complex matrix $T$, $T^* = -T$, such that

$$HM = T + D.$$

Proof. 1) $HM = T + D \Rightarrow HM + M^* H = T + D + (T + D)^* = 2D$ because $T + T^* = 0$. $H, D$ form thus a Liapounov pair.

2) Let $HM + M^* H = 2D$, $T := HM - D \Rightarrow T^* = M^* H - D^*$, and $T + T^* = HM - D + M^* H - D = 0$.

Lemma 2. Let $A(M) \neq 0$. If $\{H_j, D_j\}, j = 1, \ldots, m$, are Liapounov pairs with respect to $M$, then if $H_j, \ldots, H_m$ are linearly independent in $\mathbb{R}$, so too are $D_1, \ldots, D_m$.

Proof. Let us assume that $D_1, \ldots, D_m$ are linearly dependent, i.e. there are $a_1, \ldots, a_m \in \mathbb{R}$, $(a_1, \ldots, a_m) \neq (0, \ldots, 0)$, such that $\sum_{j=1}^m a_j D_j = 0$. One then has

$$\sum_{j=1}^m a_j D_j = \sum_{j=1}^m a_j (H_j M + M^* H_j) = \left( \sum_{j=1}^m a_j H_j \right) M + M^* \left( \sum_{j=1}^m a_j H_j \right) = 0.$$  According to the remark following Theorem 1 this means, however, that the $H_j$ are linearly dependent. Lemma 2 has thus been proved.

We will now confine our attention to matrices $M$ in upper Hessenberg form with non-zero elements in their lower co-diagonal. As explained in [1], p. 1 f, this is not an essential constraint.

In the following we shall reduce solution of the Liapounov Eq. (1) to repeated solution of the equation

$$HM = T + D, \quad T^* = -T, \quad D^* = D \text{ diagonal.} \quad (2)$$

For this purpose the solvability of Eq. (2) has to be studied more closely.

Lemma 3. Let $M$ be an upper Hessenberg matrix with non-zero elements in the lower co-diagonal, and let $h := (a_1, a_2 + i b_2, \ldots, a_n + i b_n)$. There then exists a unique complex matrix $A$ with the following properties:

1. $A$ contains $h$ as the first row
2. $A = B + i C$, $B^* = B$, $C = \text{diag} (0, c_2, \ldots, c_n) \in L(\mathbb{R}^n)$.
3. $AM = T + D$, $T^* = -T$, $D^* = D \text{ diagonal matrix.}$

Proof. Let $A = (a_{ij}), B = (b_{ij}), AM = (a_{mj})$. As the first row of $A$ is given, we can calculate the first row of $AM = T + D$. Since $B$ is Hermitian and $T$ anti-Hermitian, the first columns of $A$ and $AM$ are then known, hence $a_{21}$ and $am_{21}$ in particular. $a_{22}$ is now calculated from

$$a_{21} m_{21} + a_{22} m_{21} = am_{21} = -\overline{am_{12}}.$$