SOME REMARKS CONCERNING THE INDIVIDUAL
ERGODIC THEOREM OF INFORMATION THEORY

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UDC 517.9

Let \((X, \mu, T)\) be an ergodic dynamic system and let \(\xi = (C_1, C_2, \ldots)\) be a discrete decomposition of \(X\). Conditions are considered for the existence almost everywhere of

\[
\lim_{n \to \infty} \frac{1}{n} \log \mu (C_{\xi^n}(x)),
\]

where \(C_{\xi^n}(x)\) is the element of the decomposition \(\xi^n = \xi \vee T \xi \vee \ldots \vee T^{n-1} \xi\) containing \(x\). It is proved that the condition \(H(\xi) < \infty\) is close to being necessary. If \(T\) is a Markov automorphism and \(\xi\) is the decomposition into states, then the limit exists, even if \(H(\xi) = \infty\), and is equal to the entropy of the chain.

Definitions of all concepts used in this work can be found in [1].

Breiman [2] has proved that if \(\mu(X) = 1\) and \(\xi\) is a finite decomposition, then

\[
\lim_{n \to \infty} \frac{1}{n} \log \mu (C_{\xi^n}(x)) = h(T, \xi)
\]

exists almost everywhere. K. L. Chung [3] replaced the condition that \(\xi\) be finite by the weaker condition \(H(\xi) < \infty\). We also note the work done by Ionescu-Tulcea [4] on the same subject. In order to study the precision of the condition \(H(\xi) < \infty\), we construct an example. In this example \(T\) is a K-automorphism, \(\eta\) is a countable decomposition, \(h(T) < \infty\),

\[
H(\eta) = \infty,
\]

but

\[
\sum_{C_{\eta^k}(C_\eta)} |\log \mu (C_{\eta^k}(x))|^q < \infty
\]

for any \(q\) satisfying \(0 < q < 1\).

Example. Consider the Bernoulli scheme with a countable number of states. The set of trajectories passing through the \(k\)-th state at time zero will be denoted by \(C_k\) \((k = 1, 2, \ldots)\). Let \(\mu\) be an invariant measure on the space of trajectories of the Bernoulli scheme and let \(\mu(C_k) = 2^{-k}\). The sets \(\{C_k\} (k = 1, 2, \ldots)\) form a decomposition \(\xi\) of the trajectory space. In our discussion of this example, \(\xi\) always denotes this decomposition. As elements of the decomposition \(\eta\) we take the sets \(\{C_k \cap B_k\}\), where \(B_k\) is any element of the decomposition

\[
T^{-1} \xi \vee T^{-2} \xi \vee \ldots \vee T^{-k+1} \xi \quad (k = 1, 2, \ldots).
\]

Let

\[
L_k = \{x: x \subset T^k \cup \cup_{l > \log k \xi log k} C_l\}.
\]
We write \([a]\) for the integral part of the positive number \(a\). If \(x \in L_k\), then

\[T^{-k}x = \bigcup_{i=\log(k \log k)} C_i,\]

and so

\[C_k(T^{-k}x) \subset C_{\log(k \log k)}(T^{-k}x).\]

Since

\[2^{-\log(k \log k)} \geq k \log k \geq \lfloor k \log k \rfloor,
\]

we have

\[C_{T^k \Gamma_k}(x) \subset C_{T^k \Gamma_k} \cap \bigcup_{i=\log(k \log k)} T^{-i}(x),\]

or

\[C_{T^k \Gamma_k}(x) = C_{T^k \Gamma_k} \cap \bigcup_{i=\log(k \log k)} T^{-i}(x).\]

It follows that, if \(x \in L_k\), we have

\[\mu(C_{T^k \Gamma_k}(x)) < \mu(C_{T^k \Gamma_k} \cap T^{-i}(x)).\]

The events \(L_k\) are independent and

\[\sum_{k=1}^{\infty} \mu(L_k) = \sum_{k=1}^{\infty} 2^{-\log(k \log k)} = \infty.
\]

Hence almost all \(x\) belong to an infinite number of \(L_k\), i.e., with almost every \(x\) we can associate an increasing sequence of positive integers \(\{k_i\}\) \((i = 1, 2, \ldots)\), such that \(x \in L_{k_i}\) for all \(i\). Thus

\[\lim_{n \to \infty} \frac{1}{n} \log \mu(C_{\Gamma_{\eta}}(x)) \geq \lim_{k \to \infty} \frac{1}{k \log k} \log \mu(C_{\Gamma_{\eta+1}}(x))\]

\[= \lim_{k \to \infty} \frac{1}{k \log k} \log \mu(C_{\Gamma_{\eta+1}}(x)) = \lim_{k \to \infty} \frac{1}{k \log k} \log \mu(C_{\Gamma_{\eta+1}}(x)) = H(\xi) = \infty.
\]

The entropy of the automorphism is \(H(\xi) < \infty\); hence the assertion of the Shannon-McMillan-Breiman theorem is not true for our example. Chung's result implies that \(H(\eta) = \infty\). Hence we have the following

**Proposition.** For any \(q\) \((0 < q < 1)\) we have

\[\sum_{C_x} \mu(C_x) |\log \mu(C_x)|^q < \infty.
\]

**Proof.** For convenience in the reasoning we introduce the following

**Definition.** Let \(0 < q < 1\), and let \(\xi\) and \(\eta\) be discrete decompositions. Let

\[H_q(\eta/\xi) = \sum_{C_x} \mu(C_x) \sum_{C_y} \frac{\mu(C_x \cap C_y)}{\mu(C_y)} |\log \frac{\mu(C_x \cap C_y)}{\mu(C_y)}|^q.
\]

We write \(H_q(\eta)\) instead of \(H_q(\eta/\nu)\), where \(\nu\) is a trivial decomposition. We divide up our proof into a series of lemmas.