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Local Symmetries and Conservation Laws*

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Abstract. Starting with Lie's classical theory, we carefully explain the basic notions of the higher symmetries theory for arbitrary systems of partial differential equations as well as the necessary calculation procedures. Roughly speaking, we explain what analogs of 'higher KdV equations' are for an arbitrary system of partial differential equations and also how one can find and use them. The cohomological nature of conservation laws is shown and some basic results are exposed which allow one to calculate, in principle, all conservation laws for a given system of partial differential equations. In particular, it is shown that 'symmetry' and 'conservation law' are, in some sense, the 'dual' conceptions which coincides in the 'self-dual' case, namely, for Euler–Lagrange equations. Training examples are also given.

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Contents

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1. Introduction

In this paper the word symmetry is used in the most general sense, i.e., as a transformation into itself of an object, situation, or point of view. In mathematics, symmetries of specific structures have their own names, e.g., diffeomorphism, isometry, automorphism, etc. Symmetries of systems of nonlinear differential equations (d.e.) studied in this paper do not have a special name of their own and we won't suggest one. When a system of d.e. describes some natural phenomenon, the symmetries of this system represent the mathematical description of symmetries of this phenomenon. For this reason we find it undesirable to introduce a new term.

Symmetry considerations belong to the most universal and astonishing methods by which *homo sapiens*, starting with the invention of the wheel, successfully solves his problems in art, religion, engineering, philosophy, and science. We hope that the reader shares our opinion about the validity of this flashy phrase. We have allowed it for two reasons: firstly, to stress once again the fundamental role which the idea of symmetry plays in most general circumstances, and secondly, to state that methods of the theory of symmetries in the context of systems of nonlinear d.e. are clearly underdeveloped. This situation has its justifications, the most important of them being, obviously, the fact that, as we shall show later, the notion of a symmetry of a system of d.e. is in itself not self-evident. This notion requires for its definition a rather complicated mathematical technique which, in turn, imposes some interdisciplinary barriers with all the resulting consequences. Similarity and dimensional analysis, the Noether theorem, separation of variables and the Fourier transform, probably comprise the list of all symmetry considerations used regularly in practice in the study of concrete systems of d.e. Clearly, one wishes for more.

Every consistent theory of symmetries ought to answer the following questions:

(1) What exactly is the notion of a symmetry in a given situation?

(2) How does one find symmetries in concrete examples?

(3) What use can one make from the symmetries one has found?

In this paper we consider these questions for systems of nonlinear d.e., with the stress on the first two. A reasonably complete answer for the third question must wait for future developments.

The notion of a conservation law for a given system of nonlinear d.e. is, undoubtedly, fundamental. Physicists use the phrase 'conserved currents' in the same sense. For ordinary d.e., the notion of a conservation law is identical to the notion of an integral. For the Euler–Lagrange equations which follow from variational principles, the notion of conservation law is closely connected with the notion of the symmetry and this connection is given by the famous Noether theorem.