PROJECTIVE INJECTIONS OF GEOMETRIES AND THEIR AFFINE EXTENSIONS

Francis Buekenhout, Michel Dehon and Isabelle De Schutter

We present a general method allowing the construction geometries whose diagram is an extension of the diagram of a given geometry. Some applications of this construction process are described.

1. INTRODUCTION

We present a general method allowing the construction of many geometries with specified diagrams. Our starting point is the paper DEHON [4] in which CAYLEY programmes are developed in order to classify all flag-transitive geometries on which a given group $G$ is acting.

These programmes have been running in cases where $G$ is the affine group of an affine plane of order $q = 3, 4$ or $5$. Hundreds of geometries of rank $3$ were produced. Inspection of these lists showed among other things that a diagram of type

$$\begin{array}{c}
\circ \quad \circ \quad c \\
q - 1 \quad 1 \quad q - 1
\end{array}$$

was present in all cases. Apparently, CAYLEY (see CANNON [3]) was giving us, so far undetected models of such geometries (see BUEKENHOUT-PASINI [2] for a survey of work on this and other diagrams, also PASINI [5]). A direct construction, not using CAYLEY, soon became obvious and led us to generalizations based on the following idea.

Let $\Gamma$ be a geometry over some diagram $\Delta$ with a distinguished type of elements say $1$. Let $\Gamma_1$ be the set of elements of type $1$. Assume that $A$ is an affine space and let $P$ be its
projective space at infinity. Let an injection (that is, simply a mapping) \( \alpha \) from \( \Gamma_1 \) to the set of points of \( P \) be given. Then we construct a "geometric" \( \Gamma' \) depending on \( \Gamma, A \) and \( \alpha \), whose set of "points" is the pointset of \( A \), with point residues isomorphic to \( \Gamma \) and with a diagram extending \( \Delta \). Of course, this is too general to work well but the conditions to make the process work are astonishingly mild.

2. DEFINITIONS

An incidence structure is a set \( \Gamma = \{ \Gamma_i \mid i \in I \} \) of sets provided with an incidence relation \( \sim \) defined on \( \bigcup \Gamma_i \), such that \( x \not\sim y \) for all \( x, y \in \Gamma_i \), any \( i \). The elements of \( \Gamma_i \) are called the \( i \)-varieties or the varieties of type \( i \) of \( \Gamma \) and \( r = |I| \), which is supposed to be finite here, is the rank of \( \Gamma \).

A flag of \( \Gamma \) is a set \( F \) of pairwise incident varieties; the type \( t(F) \) of \( F \) is the set \( \{ i \mid \Gamma_i \cap F \neq \emptyset \} \) and its rank is \( |F| \). A variety \( z \) of \( \Gamma \) is said to be incident to a flag \( F \) if \( x \sim y \) for every \( y \in F \). The residue of a flag \( F \) of \( \Gamma \) is the incidence structure \( \Gamma(F) = \{ \Gamma_j(F) \mid j \in I - t(F) \} \) where \( \Gamma_j(F) \) is the set of all varieties of \( \Gamma_j \) which are incident to \( F \) and the incidence is the restriction of the incidence of \( \Gamma \) to \( \Gamma(F) \). A flag containing an element of each \( V \) of \( F \) is called a chamber.

An incidence structure \( \Gamma \) is a geometry if it satisfies the following axioms:

(i) every flag which is not a chamber is contained in at least two chambers;

(ii) for every flag \( F \) of rank at most \( r - 2 \), the incidence graph of \( \Gamma(F) \) is connected.

The diagram \( \Delta \) of a geometry \( \Gamma \) is the graph whose vertices are the elements of the set \( I \) of indices of \( \Gamma \) and whose edges are the pairs \( \{i,j\} \) such that there exists a flag \( F \) of \( \Gamma \) of type \( I - \{i,j\} \) and two non incident varieties \( x \in \Gamma_i(F) \) and \( y \in \Gamma_j(F) \).

A geometry \( \Gamma \) is flag-transitive if it possesses an automorphism group preserving each type and acting transitively on its chambers. Such a group does also act transitively on the set of all flags having a given type. So, if \( \Gamma \) is flag-transitive, for every \( i \in I \) the number of \( i \)-varieties incident to a flag \( F \) of type \( I - \{i\} \) does not depend on the choice of \( F \); this number, diminished by 1, is called the \( i \)-order of \( \Gamma \). Similarly, for every \( i,j \in I \), all the rank 2 residues of flags of type \( I - \{i,j\} \) are isomorphic and are themselves flag-transitive geometries of rank 2. So the diagram of a flag-transitive geometry can be completed by indicating, on each vertex, the order of the corresponding type of \( \Gamma \) and by labelling each edge with some information to describe the structure of the corresponding rank 2 residues of \( \Gamma \); we use the notation defined in [1] for this.