THE PROBLEM OF NON-TRIVIAL ISOMETRIES OF SURFACES
PRESERVING PRINCIPAL CURVATURES

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In this paper we deal with the Bonnet problem of determining the surfaces in the Euclidean
three dimensional space which can admit at least one nontrivial isometry that preserves the
principal curvatures (Bonnet surfaces). The problem is considered locally and examined in
the general case. The main results are: (a) Necessary and sufficient condition for a surface to
be a Bonnet surface is that it admits a special isothermal parameter system. (b) Complete
solution of the problem in the class of the isothermic surfaces. Moreover: These results
and the methods used provide a new efficient and elegant manner of proving the, already
known, fact that all helicoidal surfaces are Bonnet surfaces and determine the already known
developable Bonnet surfaces.

1. INTRODUCTION

1.1. In [1] Bonnet examined the problem of determining the surfaces real and complex
which can admit at least one non trivial isometry which preserves the principal curvatures.
He obtained some important results and he pointed out that an explicit solution of the
problem did not seem possible in the general case.

In the above problem (Bonnet problem), we assume that the lines of curvature are not
preserved because when they are also preserved, the surfaces remain the same (up to the
isometries of the ambicient space).

Any surface which is a solution of this problem is called Bonnet surface (B-surface). If two
surfaces are nontrivially isometric to each other with preservation of the principal curvatures,
we say that they are associates to each other. The set of all such surfaces is called a B-family.
The lines of curvature of an associate surface $S''$ of a surface $S$ correspond to an orthogonal
net on $S$ and vice versa. These orthogonal nets are called B-nets.
Bonnet showed that, in the general case, there is one and only one associate surface of a \(B\)-surface and thus any \(B\)-surface has one and only one \(B\)-net. He also showed that \(B\)-surfaces having more than one \(B\)-nets are isothermic Weingarten (\(W\)-) surfaces which are isometric to a surface of revolution and have infinite number (\(\infty^1\)) of \(B\)-nets.

After Bonnet, many researchers studied the problem again (see \([2\text{-}13]\)). They considered the problem locally but not for the general case and mainly for the special case of \(B\)-surfaces having \(\infty^1\) \(B\)-nets (i.e. for the isothermic \(B\)-surfaces). It would seem plausible that if a surface has \(\infty^1\) \(B\)-nets, then the corresponding \(B\)-family would have \(\infty^1\) members. E. Cartan \([3]\) has shown that this result is not true. Cartan, in his long and detailed study, examined the problem for the isothermic surfaces which are real and analytic and he showed that: (1) for only surfaces of constant mean curvature, other than planes and spheres, their \(B\)-families have \(\infty^1\) members. Here, we must say that for the right circular cylinders their \(B\)-families consist of one member which can be realized by \(\infty^1\) nontrivial isometries preserving the principal curvatures. (2) for the isothermic surfaces of nonconstant mean curvature the \(B\)-families may have only 1 member or only 2 members or only 3 members up to congruences and symmetries. But a small correction should be made in the above result (2) which was considered "unexpected" by Cartan. In fact, Cartan states that two helicoids belonging to the "\(B\)-family of 3 elements" are symmetrical and so they are not considered as different. However, the principal curvatures of the symmetrical surfaces have the same absolute value but their signs are opposite, that is, these surfaces can not be members of the same \(B\)-family. In the following we shown that these helicoids are not symmetrical; they are two different helicoids \(S\) and \(S'\) of the parameter \(a\) and \(-a\) respectively and of the same generated meridian curve. So the mentioned \(B\)-family consists of 4 members, not 3. Furthermore, we explained the reasons of the above unexpected result geometrically.

1.2. In this paper we consider the Bonnet problem in the general case for real and sufficiently differentiable surfaces, and we give the characterization of the \(B\)-surfaces by the fundamental theorem below. As an application of this we consider the special case where the surface is isothermic and we obtain the results, which were found after rather long calculations by Cartan in \([3]\), in a straightforward and efficient manner. Moreover, we determine all developable \(B\)-surfaces \([8,11]\) and show that all the helicoids are \(B\)-surfaces \([9]\).

In the following the coefficients of the first and second fundamental forms of the surface \(S\) will be denoted by \(E, F, G\) and \(L, M, N\) respectively and the definitions

\[
H = (r + r^*)/2, \; K = rr^*, \; J = (r - r^*)/2
\]

will be used where \(r\) and \(r^*\) are the principal curvatures of \(S\). The corresponding quantities of the associate surface \(S'\) will be indicated by the prime symbols.