Monotony in Interpolatory Quadratures

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Summary. Let \( f(z) = \sum_{j=0}^{\infty} t_{2j}z^{2j}, \ t_{2j} \geq 0 \ (j=0, 1, 2, \ldots) \), be holomorphic in an open disc with the centre \( z_0 = 0 \) and radius \( r>1 \). Let \( Q_n \ (n=1, 2, \ldots) \) be interpolatory quadrature formulas approximating the integral \( \int_{-1}^{+1} f(x)dx \). In this paper some classes of interpolatory quadratures are considered, which are based on the zeros of orthogonal polynomials corresponding to an even weight function. It is shown that the sequences \( Q_n[f] \ (n=1, 2, \ldots) \) are monotone. Especially we will prove monotony in Filippi's quadrature rule and with an additional assumption on \( f \) monotony in the Clenshaw-Curtis quadrature rule.

Subject classifications: AMS(MOS): 65 D 30; CR: 5.16.

1. Introduction

The integral \( \int_{-1}^{+1} f(x)dx \) is often approximated by a linear functional \( Q_n \) with

\[
Q_n[f] = \sum_{v=1}^{n} a_{v,n} f(x_{v,n}), \quad -1 \leq x_{n,n} < x_{n-1,n} < \ldots < x_{2,n} < x_{1,n} \leq +1.
\]

Such a functional is called a quadrature formula. If we assume that

\[
Q_n[p] = \int_{-1}^{+1} p(x)dx
\]

for every polynomial \( p \) of degree at most \( n-1 \), then \( Q_n \) is called an interpolatory quadrature formula. It is uniquely determined by the abscissas \( x_{v,n} \ (v=1, 2, \ldots, n) \). A system of abscissas \( x_{v,n} \ (v=1, 2, \ldots, n; n=1, 2, \ldots) \) defines a quadrature rule, that is a sequence of quadrature formulas.
Now let \( w \) be a weight function on \((-1, +1)\) and \( p_n(x) (n=0, 1, 2, \ldots) \) the corresponding sequence of orthonormal polynomials with positive leading coefficients \( k_n \). In this paper I will consider the following two types of interpolatory quadratures:

- the abcissas \( x_{v,n} \) are the zeros of \( p_n(x) \),
- the abcissas \( x_{v,n} \) are the zeros of \( q_n(x) = (x^2 - 1)p_{n-2}(x) \).

The Gauss-(\( w(x) = 1 \)) and the Filippi-quadrature rule (\( w(x) = \sqrt{1-x^2} \)) are examples of the first type, Lobatto-(\( w(x) = 1 - x^2 \)) and Clenshaw-Curtis-quadrature rule (\( w(x) = \sqrt{1-x^2} \)) are examples of the second type of interpolatory quadratures.

If a given quadrature rule converges for some set \( K \) of functions, that means
\[
\lim_{n \to \infty} Q_n[f] = \int_{-1}^{+1} f(x) \, dx \quad \text{for } f \in K,
\]
it is interesting to investigate the monotony of the sequence \( Q_n[f] (n=1, 2, \ldots) \) for a given \( f \in K \), because monotony of this sequence means that for \( m > n \)
\[
Q_m[f] \text{ will be a better approximation to } \int_{-1}^{+1} f(x) \, dx \text{ than } Q_n[f].
\]

F. Strenger [7] and H. Braß [3] have studied the problem of monotony in the Gauss quadrature rule. It seems that this is the only case of interpolatory quadrature based on the zeros of orthogonal polynomials which has been treated so far. The aim of this paper is to prove some results on the monotony in further interpolatory quadratures.

2. The Results

Let \( f(z) = \sum_{j=0}^{\infty} t_j z^j \) be holomorphic in the open disc with the centre \( z_0 = 0 \) and radius \( r > 1 \).

Further let
\[
R_n[f] = \int_{-1}^{+1} f(x) \, dx - Q_n[f]
\]
be the error functional corresponding to \( Q_n \). Then we make the following general assumptions

(i) \( \sup_n \| Q_n \| < \infty \),

(ii) \( w(x) = w(-x) \),

(iii) \( t_{2j} \geq 0, \quad j = 0, 1, 2, \ldots \).

From (ii) it follows, that \( R_n[f] = 0 \) for every odd function; thus it suffices to consider even functions respectively the even parts of functions. (i) implies not